

STATISTICS FOR S_n ACTING ON k -SETS

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ABSTRACT. We study the natural action of S_n on the set of k -subsets of the set $\{1, \dots, n\}$ when $1 \leq k \leq \frac{n}{2}$. For this action we calculate the maximum size of a minimal base, the height and the maximum length of an irredundant base.

Here a *base* is a set with trivial pointwise stabilizer, the *height* is the maximum size of a subset with the property that its pointwise stabilizer is not equal to the pointwise stabilizer of any proper subset, and an *irredundant base* can be thought of as a chain of (pointwise) set-stabilizers for which all containments are proper.

1. INTRODUCTION

{s: intro}

In this note we study three statistics pertaining to primitive permutation groups. Our main theorem gives the value of these three statistics for the permutation groups S_n acting (in the natural way) on the set of k -subsets of the set $\{1, \dots, n\}$.

Before we state our main result, let us briefly define the three statistics in question (more complete definitions, as well as some background information, are given in §1.1): suppose that G is a finite permutation group on a set Ω . We define, first, $B(G, \Omega)$ to be the maximum size of a minimal base for the action of G ; we define, second, $H(G, \Omega)$ to be the maximum size of a subset $\Lambda \subseteq \Omega$ that has the property that its pointwise stabilizer is not equal to the pointwise stabilizer of any proper subset of Λ ; we define, third, $I(G, \Omega)$ to be the maximum length of an irredundant base for the action of G .

Our main result is the following.

{t: main}

Theorem 1.1. *Let k and n be positive integers with $1 \leq k \leq \frac{n}{2}$. Consider S_n acting in the natural way on Ω_k , the set of k -subsets of $\{1, \dots, n\}$.*

$$(1) \quad I(S_n, \Omega_k) = \begin{cases} n-1, & \text{if } \gcd(n, k) = 1; \\ n-2, & \text{otherwise.} \end{cases}$$

$$(2) \quad B(S_n, \Omega_k) = H(S_n, \Omega_k) = \begin{cases} n-1, & \text{if } k = 1; \\ n-2, & \text{if } k = 2 \text{ or} \\ & \text{if } k \geq 3 \text{ and } n = 2k + 2; \\ n-3, & \text{otherwise.} \end{cases}$$

{s: defs}

1.1. Definition of statistics. Throughout, consider a finite permutation group G on a set Ω . Let $\Lambda = \{\omega_1, \dots, \omega_k\} \subseteq \Omega$; we write $G_{(\Lambda)}$ or $G_{\omega_1, \omega_2, \dots, \omega_k}$ for the pointwise stabilizer.

If $G_{(\Lambda)} = \{1\}$, then we say that Λ is a *base*. We say that a base is a *minimal base* if no proper subset of it is a base. We denote the minimum size of a minimal base $b(G, \Omega)$, and the maximum size of a minimal base $B(G, \Omega)$.

We say that Λ is an *independent set* if its pointwise stabilizer is not equal to the pointwise stabilizer of any proper subset of Λ . We define the *height* of G to be the maximum size of an independent set, and we denote this quantity $H(G, \Omega)$.

Given an ordered sequence of elements of Ω , $[\omega_1, \omega_2, \dots, \omega_\ell]$, we can study the associated *stabilizer chain*:

$$G \geq G_{\omega_1} \geq G_{\omega_1, \omega_2} \geq G_{\omega_1, \omega_2, \omega_3} \geq \dots \geq G_{\omega_1, \omega_2, \dots, \omega_\ell}.$$

If all the inclusions given above are strict, then the stabilizer chain is called *irredundant*. If, furthermore, the group $G_{\omega_1, \omega_2, \dots, \omega_\ell}$ is trivial, then the sequence $[\omega_1, \omega_2, \dots, \omega_\ell]$ is called an *irredundant base*. The length of the longest possible irredundant base is denoted $I(G, \Omega)$. Note that, defined in this way, an irredundant base is not a base (because it is an ordered sequence, not a set).

Let us make some basic observations. First, it is easy to verify the following inequalities:

$$(1.1) \quad \{\text{basic}\} \quad b(G, \Omega) \leq B(G, \Omega) \leq H(G, \Omega) \leq I(G, \Omega).$$

Key words and phrases. permutation group; height of a permutation group; relational complexity; base size.

Second, it is easy to see that $\Lambda = \{\omega_1, \omega_2, \dots, \omega_k\}$ is independent if and only if the pointwise stabilizer of Λ is not equal to the pointwise stabilizer of $\Lambda \setminus \{\omega_i\}$ for all $i = 1, \dots, k$. Third, any subset of an independent set is independent.

{s: context}

1.2. Some context. Our interest in the statistics considered here was stimulated by our study of yet another statistic, the *relational complexity* of the permutation group G , denoted $\text{RC}(G, \Omega)$. This statistic was introduced in [CMS96]; it can be defined as the least k for which G can be viewed as the automorphism group of a homogeneous relational structure whose relations are k -ary [Che16].

It is an exercise to confirm that $\text{RC}(G, \Omega) \leq \text{H}(G, \Omega) + 1$ for any permutation group G on a set Ω [GLS21]. Anecdotaly it would seem that $\text{RC}(G, \Omega)$ tends to track $\text{H}(G, \Omega) + 1$ rather closely: it often seems to equal this value or to be rather close to it. In this respect Theorem 1.1 tells us that the action of S_n on the set of k -sets is an aberration: in [Che00], Cherlin calculates that $\text{RC}(S_n, \Omega_k) = \lfloor \log_2 k \rfloor + 2$; asymptotically this is very far from the value for the height that is given in Theorem 1.1.

In a different direction, an earlier result with Spiga, along with work of Kelsey and Roney-Dougall, asserts that the statistics $\text{H}(G, \Omega)$ and $\text{I}(G, \Omega)$ satisfy a particular upper bound whenever G is primitive and not in a certain explicit family of permutation groups [GLS21, KRD]. Ultimately we would like to calculate the value of $\text{H}(G, \Omega)$ and $\text{I}(G, \Omega)$ for all of the permutation groups in this explicit family; our calculation of $\text{H}(S_n, \Omega_k)$ and $\text{I}(S_n, \Omega_k)$ is the first step in this process.

The one statistic that we have neglected in our study is $\text{b}(G, \Omega)$. The value of this statistic for the actions under consideration has not been completely worked out, although significant progress has been made (see [CGG⁺13, Hal12] as well as [BC11] and the references therein). On the other hand, for those primitive actions of S_n for which a point-stabilizer acts primitively in the natural action on $\{1, \dots, n\}$, the value of $\text{b}(G, \Omega)$ is known [BGS11].

Finally it is worth mentioning that, in general, $\ell(G)$, the maximum length of a chain of subgroups in a group G , is an upper bound for $\text{I}(G, \Omega)$ for any faithful action of the group G on a set Ω . It is known that $\ell(S_n) = \lfloor \frac{3n-1}{2} \rfloor - b_n$, where b_n is the number of 1's in the binary expansion of n [CST89].

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2. THE PROOF

In this section we prove Theorem 1.1. Throughout the proof we will write G for S_n . We need some terminology.

Suppose that $\Delta = \{\delta_1, \dots, \delta_\ell\}$ is a set of non-empty subsets of $\{1, \dots, n\}$. We define \mathcal{P}_Δ , the *partition associated with Δ on $\{1, \dots, n\}$* , to be the partition of $\{1, \dots, n\}$ associated with the equivalence relation \sim given as follows: for $x, y \in \{1, \dots, n\}$, we have $x \sim y$ if and only if for all $i = 1, \dots, \ell$, $x \in \delta_i \iff y \in \delta_i$. If Δ is empty, then we define \mathcal{P}_Δ to be the partition with a single part of size n .

It is an easy exercise to check that, first, \mathcal{P}_Δ can be obtained by taking intersections of all elements of Δ ; second, the pointwise stabilizer of Δ in S_n is simply the stabilizer of all parts of \mathcal{P}_Δ .

If $i, j \in \{1, \dots, n\}$ and $\omega \in \Omega_k$, then we will say that ω *splits* i and j if $|\{i, j\} \cap \omega| = 1$. In particular, if ω splits i and j then for any set Δ such that $\omega \in \Delta \subseteq \Omega_k$, we have $i \not\sim j$ where \sim is the equivalence relation associated with Δ .

{s:i}

2.1. The result for $\text{I}(G, \Omega_k)$. We will use the terminology above and begin with a couple of lemmas.

{1: i}

Lemma 2.1. *Let $d, e \in \mathbb{Z}^+ \cup \{0\}$ with $e > d$, let H be a permutation group on the set $\{1, \dots, n\}$, let $\Delta = \{\delta_1, \dots, \delta_d\}$ be a set of non-empty subsets of $\{1, \dots, n\}$, let Λ be a set of non-empty subsets of $\{1, \dots, n\}$ that contains Δ and let $\Lambda \setminus \Delta = \{\lambda_{d+1}, \dots, \lambda_e\}$. Suppose that*

$$H \succeq H_{\delta_1} \succeq H_{\delta_1, \delta_2} \succeq \dots \succeq H_{\delta_1, \dots, \delta_d} \succeq H_{\delta_1, \dots, \delta_d, \lambda_{d+1}} \succeq H_{\delta_1, \dots, \delta_d, \lambda_{d+1}, \lambda_{d+2}} \succeq \dots \succeq H_{(\Lambda)}.$$

If \mathcal{P}_Δ has r parts and \mathcal{P}_Λ has s parts, then $|\Lambda| = e \leq d + s - r$.

Note that if Δ is empty, then the lemma applies with $d = 0$ and $r = 1$, and we obtain that $|\Lambda| \leq s - 1$.

Proof. For $i = 1, \dots, d$, let \mathcal{P}_i be the partition associated with the set $\{\delta_1, \dots, \delta_i\}$ and, for $i = d+1, \dots, e$, let \mathcal{P}_i be the partition associated with the set $\Delta \cup \{\lambda_{d+1}, \lambda_{d+2}, \dots, \lambda_i\}$. Since all the containments are proper, \mathcal{P}_{i+1} has at least one more part than \mathcal{P}_i for all $i = 1, \dots, e-1$. There are $|\Delta|$ containments up to $H_{(\Delta)} = H_{\delta_1, \dots, \delta_d}$ and then the number of containments after that is at most $s - r$. The result follows. \square

{1: j}

Lemma 2.2. *Let $\ell \in \mathbb{Z}^+$, let $g = \text{gcd}(n, k)$, let $\omega_1, \dots, \omega_\ell$ be k -subsets of Ω and let \mathcal{P}_i be the partition associated with $\{\omega_1, \dots, \omega_i\}$. If \mathcal{P}_{i+1} has exactly one more part than \mathcal{P}_i for all $i = 1, \dots, \ell - 1$, then all parts of \mathcal{P}_ℓ have size divisible by g .*

Proof. We proceed by induction. Observe that \mathcal{P}_1 has two parts, one of size k and the other of size $n - k$. Both k and $n - k$ are divisible by g and so the result is true for $i = 1$.

Let $i \in \{1, \dots, k - 1\}$ and assume that all parts of \mathcal{P}_i have size divisible by g . The property that \mathcal{P}_{i+1} has exactly one more part than \mathcal{P}_i implies that, with precisely one exception, if P is a part of \mathcal{P}_{i+1} , then $|\omega_{i+1} \cap P| \in \{0, |P|\}$. In other words

$$\omega_{i+1} = P_1 \cup \dots \cup P_m \cup X,$$

where m is some integer, P_1, \dots, P_m are parts of \mathcal{P}_i and X is a proper subset of part P_{m+1} . Note that, by assumption, $|\omega_{i+1}| = k$ is divisible by g . What is more the inductive hypothesis implies that $|P_1|, \dots, |P_m|$ are divisible by g , hence the same is true of $|X|$. But now \mathcal{P}_{i+1} has the same parts as \mathcal{P}_i except that part P_{m+1} has been replaced by two parts, X and $P_{m+1} \setminus X$, both of which have size divisible by g . The result follows. \square

We are ready to prove item (1) of Theorem 1.1. First we let Λ be an independent set and observe that Lemma 2.1 implies that $|\Lambda| \leq n - 1$; thus $I(G, \Omega_k) \leq n - 1$.

Next we suppose that $\gcd(n, k) = 1$ and we must show that there exists a stabilizer chain

$$G \supseteq G_{\omega_1} \supseteq G_{\omega_1, \omega_2} \supseteq G_{\omega_1, \omega_2, \omega_3} \supseteq \dots \supseteq G_{\omega_1, \omega_2, \dots, \omega_{n-1}},$$

with $\omega_1, \dots, \omega_{n-1} \in \Omega_k$. Observe that if such a chain exists, then, writing \mathcal{P}_i for the partition associated with $\{\omega_1, \dots, \omega_i\}$, it is clear that, for $i = 1, \dots, n - 2$, the partition \mathcal{P}_{i+1} has exactly one more part than \mathcal{P}_i .

We show the existence of such a chain by induction on k : If $k = 1$, then the result is obvious. Write $d = \lfloor n/k \rfloor$ and write $n = dk + r$. For $i = 1, \dots, d$, we set

$$\omega_i = \{(i - 1)k + 1, \dots, ik\}.$$

The stabilizer of these d sets is associated with a partition, \mathcal{P}_d , of d parts of size k and one of size r . Now we will choose the next sets, $\omega_{d+1}, \dots, \omega_{d+k-1}$, so that they all contain the part of size r and so that the remaining $k - r$ points in each are elements of ω_1 . Observe that, since $(n, k) = 1$, we know that $(k, k - r) = 1$. Now the inductive hypothesis asserts that we can choose $k - 1$ subsets of ω_1 , all of size $k - r$, so that the corresponding stabilizer chain in $\text{Sym}(\{1, \dots, k\})$ is of length $k - 1$, i.e. so that the corresponding chain of partitions of $\{1, \dots, k\}$ has the property that each partition has exactly one more part than the previous.

We can repeat this process for $\omega_2, \dots, \omega_d$, at the end of which we have constructed a stabilizer chain of length dk for which the associated partition, \mathcal{P}_{dk} , has dk parts of size 1 and 1 part of size r . A further $r - 1$ subgroups can be added to the stabilizer chain by stabilizing sets of form

$$\{1, \dots, k - 1, dk + i\}$$

for $i = 1, \dots, r - 1$. We conclude that $I(G, \Omega_k) = n - 1$ if $\gcd(n, k) = 1$.

On the other hand, let us see that $I(G, \Omega_k) \geq n - 2$ in general. Define

$$\omega_i = \begin{cases} \{1, \dots, k - 1, k + i - 1\}, & \text{if } i = 1, \dots, n - k; \\ \{i - (n - k), k + 1, \dots, 2k - 1\} & \text{if } i = (n - k) + 1, \dots, n - 2. \end{cases}$$

It is easy to check that the corresponding stabilizer chain

$$G \supseteq G_{\omega_1} \supseteq G_{\omega_1, \omega_2} \supseteq G_{\omega_1, \omega_2, \omega_3} \supseteq \dots \supseteq G_{\omega_1, \omega_2, \dots, \omega_{n-2}}$$

is irredundant for $G = S_n$.

Finally we assume that $\gcd(n, k) = g > 1$ and we show that $I(G, \Omega_k) \leq n - 2$. We must show that it is not possible to construct a stabilizer chain of length $n - 1$; as we saw above such a chain would have the property that at every stage the corresponding partition \mathcal{P}_{i+1} has exactly one more part than \mathcal{P}_i .

Suppose that we have a stabilizer chain with the property that, for all i , \mathcal{P}_{i+1} has exactly one more part than \mathcal{P}_i . Now Lemma 2.2 implies that all parts of \mathcal{P}_i have size divisible by g , for all i . We see immediately that such a stabilizer chain is of length at most $n/g - 1 < n - 1$ and we are done.

2.2. Preliminaries for $B(G, \Omega_k)$ and $H(G, \Omega_k)$. Note, first, that for the remaining statistics the result for $k = 1$ is immediate; thus we assume from here on that $k > 1$. Note, second, that to prove what remains we need to show that there exists a lower bound for $B(G, \Omega_k)$ which equals an upper bound for $H(G, \Omega_k)$.

2.3. **The case $k = 2$.** First assume that $k = 2$, and observe that

$$\left\{ \{1, 2\}, \{1, 3\}, \dots, \{1, n-1\} \right\}$$

is a minimal base for S_n acting on Ω_2 , and we obtain the required lower bound.

For the upper bound on $H(G, \Omega_2)$ we let Λ be an independent set. We construct a graph, Γ_Λ , on the vertices $\{1, \dots, n\}$ as follows: there is an edge between i and j if and only if $\{i, j\} \in \Lambda$.

{1: ind}

Lemma 2.3. *Suppose that H is a permutation group on Ω and consider the natural action of H on Ω_2 . If Λ is an independent subset of Ω_2 with respect to the action of H , then Γ_Λ contains no loops.*

Proof. Suppose that $[i_1, \dots, i_\ell]$ is a loop in the graph, i.e. $E_j = \{i_j, i_{j+1}\}$ is in Λ for $j = 1, \dots, \ell - 1$, along with $E_\ell = \{i_1, i_\ell\}$. Now observe that if E_j is removed from Λ for some j , then the stabilizer of the resulting set, $\Lambda \setminus \{E_j\}$, fixes the two vertices contained in E_j . But this implies that Λ is not independent, a contradiction. \square

We apply Lemma 2.3 to the action of $G = S_n$ on Ω_2 and conclude that the graph Γ_Λ is a forest. If Γ_Λ is disconnected, then the result follows immediately. Assume, then, that Γ_Λ is connected, i.e. it is a tree on n vertices. In this case there are $n - 1$ edges and we calculate directly that the point-wise stabilizer of Λ is trivial. But now, observe that if we remove any set from Λ , then the point-wise stabilizer remains trivial. This is a contradiction and the result follows.

{s: lower}

2.4. **A lower bound for $B(G, \Omega_k)$ when $k > 2$.** Assume for the remainder that $k > 2$. We prove the lower bound first: first observe that the following set, of size $n - 3$, is a minimal base with respect to S_n (note that there are $n - k - 1$ sets listed on the first row, and $k - 2$ listed altogether on the second and third):

$$(2.1) \quad \left\{ \begin{array}{l} \{1, 2, \dots, k-1, k\}, \{1, 2, \dots, k-1, k+1\}, \dots, \{1, 2, \dots, k-1, n-2\}, \\ \{1, n-(k-1), n-(k-2), n-(k-3), \dots, n-1\}, \{2, n-(k-1), n-(k-2), n-(k-3), \dots, n-1\}, \\ \dots, \{k-2, n-(k-1), n-(k-2), n-(k-3), \dots, n-1\} \end{array} \right\}.$$

To complete the proof of the lower bound, we must deal with the case $n = 2k + 2$. For this we observe that the following set, which is of size $n - 2 = 2k$, is a minimal base:

$$(2.2) \quad \left\{ \{1, \dots, k+1\} \setminus \{i\} \mid i = 1, \dots, k \right\} \cup \left\{ \{k+2, \dots, 2k+2\} \setminus \{i\} \mid i = k+2, \dots, 2k+1 \right\}$$

2.5. **An upper bound for $H(G, \Omega_k)$ when $k > 2$.** We must prove that, if $n \geq 2k$, then an independent set in Ω_k has size at most $n - 3$, except when $n = 2k + 2$, in which case it has size at most $n - 2$. It turns out that it is easy to get close to this bound in a much more general setting, as follows.

{1: n-2}

Lemma 2.4. *Let $n \geq 2$, let Δ be a set of subsets of $\Omega = \{1, \dots, n\}$ and suppose that Δ is independent with respect to the action of $G = S_n$ on the power set of Ω . Then one of the following holds:*

- (1) *There exists $\delta \in \Delta$ with $|\delta| \in \{1, n-1\}$.*
- (2) *$|\Delta| \leq n - 2$.*

Proof. Let us suppose that (1) does not hold; we will prove that (2) follows. Note that if $\delta \in \Delta$ with $|\delta| > \frac{n}{2}$, then we can replace δ with $\Omega \setminus \delta$ and the resulting set will still be independent. Note too that, since Δ is independent, all sets in Δ are non-empty. Thus we can assume that $1 < |\delta| \leq \frac{n}{2}$ for all $\delta \in \Delta$.

Suppose that there exist $\delta_1, \delta_2 \in \Delta$ such that $\delta_1 \cap \delta_2 \neq \emptyset$. Since $|\delta_1|, |\delta_2| \leq \frac{n}{2}$ this means that $\Omega \setminus (\delta_1 \cup \delta_2) \neq \emptyset$. We conclude that $\mathcal{P}_{\{\delta_1, \delta_2\}}$ contains 4 parts. Now the result follows from Lemma 2.1.

Suppose, instead, that $\delta_1 \cap \delta_2 = \emptyset$ for all distinct $\delta_1, \delta_2 \in \Delta$. If $|\Delta| \geq 1$, then \mathcal{P} has at most $n - 1$ parts and the result follows from Lemma 2.1. If $|\Delta| = 0$, then the result is true since we assume that $n \geq 2$. \square

To improve the upper bound in Lemma 2.4 (2) from $n - 2$ to $n - 3$ we will need to do quite a bit of work (and we will need to deal with some exceptions). In what follows we set Λ to be an independent set in Ω_k and, to start with at least, we drop the requirement that $n \geq 2k$.

As when $k = 2$, it is convenient to think of Λ as being the set of hyperedges in a k -hypergraph, Γ_Λ , with vertex set $\Omega = \{1, \dots, n\}$. From here on we will write “edge” in place of “hyperedge”. We think of two edges as being *incident* in Γ_Λ if they intersect non-trivially. If Δ is a set of edges in this graph (i.e. $\Delta \subseteq \Lambda$), then the *span* of Δ is the set of vertices equalling the union of all edges in Δ .

Write $\Gamma_{C_1}, \dots, \Gamma_{C_\ell}$ for the connected components of Γ_Λ ; in particular ℓ is the number of connected components in Γ_Λ . For Γ_{C_i} , we write C_i for the vertex set and Λ_{C_i} for the edge set.

In what follows we repeatedly use the fact that if $\lambda_1, \dots, \lambda_j$ are elements of the independent set Λ , then we must have

$$G \succeq G_{\lambda_1} \succeq G_{\lambda_1, \lambda_2} \succeq \dots \succeq G_{\lambda_1, \lambda_2, \dots, \lambda_j}.$$

This in turn means that, for all $i = 1, \dots, j-1$, the partition $\mathcal{P}_{\lambda_1, \dots, \lambda_{i+1}}$ has more parts than $\mathcal{P}_{\lambda_1, \dots, \lambda_i}$.

Lemma 2.5.

- (1) If Γ_Λ has a connected component with exactly one edge, then $|\Lambda| \leq n-3$.
- (2) If $\ell \geq 3$, then $|\Lambda| \leq n-3$.
- (3) Suppose that $\ell = 2$, that $|\Gamma_{C_i}| \geq 2$ for $i = 1, 2$, and that there exist incident edges E_1, E_2 in Λ_{C_1} such that the span of $\{E_1, E_2\}$ is not equal to C_1 . Then $|\Lambda| \leq n-3$.

Proof. For (1), observe that \mathcal{P}_Λ has at most $n-k+1$ parts. Then Lemma 2.1 yields the result.

We may assume, then, that any connected component of Λ either contains at least 2 edges or none. To prove (2) we go through the possibilities:

- If there are at least 3 components with no edges, then, \mathcal{P}_Λ has at most $n-2$ parts and Lemma 2.1 yields the result.
- Suppose there are 2 components with no edges and at least 1 component, C_1 , containing 2 edges. Let E_1, E_2 be incident edges in Λ_{C_1} and observe that $\mathcal{P}_{\{E_1, E_2\}}$ contains 4 parts while \mathcal{P}_Λ contains at most $n-1$ parts; Lemma 2.1 yields the result.
- Suppose there are at least 3 components in total and at least 2 components, C_1 and C_2 , containing 2 edges. Let E_1, E_2 be incident edges in Λ_{C_1} , let F_1, F_2 be incident edges in Λ_{C_2} and observe that $\mathcal{P}_{\{E_1, E_2, F_1, F_2\}}$ contains 7 parts while \mathcal{P}_Λ contains at most n parts; Lemma 2.1 yields the result.

We have proved (2). For (3) let E_1, E_2 be incident edges in Λ_{C_1} for which the span of $\{E_1, E_2\}$ is not equal to C_1 , let F_1, F_2 be incident edges in Λ_{C_2} . We can see that $\Lambda \setminus (E_1 \cup E_2 \cup F_1 \cup F_2)$ is non-empty; thus $\mathcal{P}_{\{E_1, E_2, F_1, F_2\}}$ contains 7 parts and the result again follows from Lemma 2.1. \square

The next result deals with a particular case when Λ is connected.

Lemma 2.6. *If $|\Lambda| \geq 2$ and Ω is spanned by 2 incident edges, then*

$$|\Lambda| \leq \begin{cases} n-1, & \text{if } n = k+1; \\ n-2, & \text{if } n > k+1. \end{cases}$$

Proof. Since Ω is spanned by 2 incident edges, we have $k > |\Omega|/2$. Observe that, for any $\lambda \in \Lambda$, the set $\Omega \setminus \lambda$ is a subset of size $|\Omega| - k$. Since the pointwise stabilizers in $\text{Sym}(\Omega)$ of the subsets λ and $\Omega \setminus \lambda$ are equal, we obtain that

$$\overline{\Lambda} = \{\Omega \setminus \lambda \mid \lambda \in \Lambda\}$$

is an independent set with respect to the action of $\text{Sym}(\Omega)$ on Ω_j where $j = |\Omega| - k$. Since $j < |\Omega|/2 < k$ it is clear that Ω is not spanned by 2 edges in $\overline{\Lambda}$. If $j \geq 1$, then Lemma 2.4 implies that $|\overline{\Lambda}| = |\Lambda| = |\Lambda| \leq n-2$, as required. If $j = 1$, then the result is obvious. \square

From here on we impose the condition that $n \geq 2k$.

Lemma 2.7. *Suppose that $n \geq 2k$, that $\ell = 2$ and that $|\Gamma_{C_i}| \geq 2$ for $i = 1, 2$. Then*

$$|\Lambda| \leq \begin{cases} n-2, & \text{if } n = 2k+2; \\ n-3, & \text{otherwise.} \end{cases}$$

Proof. Item (3) of Lemma 2.5 yields this result in the case where there exist $i \in \{1, 2\}$ and incident edges E_1, E_2 in Λ_{C_i} such that the span of $\{E_1, E_2\}$ is not equal to C_i . Thus we may assume that, for $i = 1, 2$ and for distinct $E_1, E_2 \in \Gamma_{C_i}$, the span of $\{E_1, E_2\}$ is equal to C_i .

We claim that for each $i = 1, 2$, the set Λ_{C_i} is independent with respect to the action of $\text{Sym}(C_i)$ on C_i . To see this observe that $\Lambda = \Lambda_{C_1} \cup \Lambda_{C_2}$ and that, by definition, Λ_{C_2} must be an independent set for $H := G_{(\Lambda_{C_1})}$. But $H = H_0 \times \text{Sym}(C_2)$ where $H_0 < \text{Sym}(C_1)$. Now if $\Delta \subseteq \Lambda_{C_2}$, then $H_{(\Delta)} = H_0 \times \text{Sym}(C_1)_{(\Delta)}$. In particular, if $\Delta_1, \Delta_2 \subseteq \Lambda_{C_2}$, then $H_{(\Delta_1)} = H_{(\Delta_2)}$ if and only if $\text{Sym}(C_2)_{(\Delta_1)} = \text{Sym}(C_2)_{(\Delta_2)}$. This implies immediately that Λ_{C_2} is independent with respect to the action of $\text{Sym}(C_2)$ on C_2 , and the same argument works for C_1 .

Now we apply Lemma 2.6 to these two actions. We conclude that, for each $i = 1, 2$, either $|C_i| = k+1$ and $|\Lambda_{C_i}| \leq |C_i| - 1$, or else $|\Lambda_{C_i}| \leq |C_i| - 2$. The result now follows from the fact that $|\Lambda| = |\Lambda_{C_1}| + |\Lambda_{C_2}|$ and $n = |C_1| + |C_2|$. \square

{1: componen

{1: b2}

{1: special}

Notice that Lemma 2.7 attends to the strange appearance of “ $n = 2k + 2$ ” in the statement of item (2) of Theorem 1.1. Before we prove Theorem 1.1 we need one more lemma.

{1: final}

Lemma 2.8. *Suppose that $n \geq 2k$. Suppose that either Λ is connected, or else it has two connected components, exactly one of which is a single isolated point. Then $|\Lambda| \leq n - 3$.*

Proof. Note that the supposition, along with the fact that $n \geq 2k$, implies that Ω contains 2 incident edges and Ω is not spanned by 2 edges. Consider E_1, E_2 , a pair of incident edges in Λ . Let $\Pi = \{E_1, E_2\}$ and observe that the parts of \mathcal{P}_Π are

$$(2.3) \quad R := E_1 \cap E_2, \quad S := E_1 \setminus (E_1 \cap E_2), \quad T := E_2 \setminus (E_1 \cap E_2) \quad \text{and} \quad U := \Omega \setminus (E_1 \cup E_2);$$

in particular $|\mathcal{P}_\Pi| = 4$.

If i and j are distinct elements of Ω that are unsplit by any element of Λ , then \mathcal{P}_Λ has at most $n - 1$ parts. Then Lemma 2.1 implies that $|\Lambda| \leq n - 3$ as required. Thus we assume that all distinct elements of Ω are split by an element of Λ .

These observations imply that we can write

$$(2.4) \quad \Lambda = \{E_1, E_2\} \cup \Lambda_R \cup \Lambda_S \cup \Lambda_T \cup \Lambda_U,$$

where, for $X \in \{R, S, T, U\}$, Λ_X is the set of elements in Λ that split pairs of distinct elements in X .

If there exists $E_3 \in \Lambda$ such that $\mathcal{P}_{\{E_1, E_2, E_3\}}$ contains 6 parts, then Lemma 2.1 implies that $|\Lambda| = n - 3$ and we are done. Thus we assume that $\mathcal{P}_{\{E_1, E_2, E_3\}}$ contains 5 parts for all choices of $E_3 \in \Lambda \setminus \{E_1, E_2\}$. In particular if $E_3 \in \Lambda_X$, then E_3 does not split any pairs of elements in Λ_Y for $Y \in \{R, S, T, U\} \setminus X$. This means, first, that if $E_3 \cap Y \neq \emptyset$ for some $Y \in \{R, S, T, U\} \setminus X$, then $E_3 \supset Y$; it means, second, that the sets $\Lambda_R, \Lambda_S, \Lambda_T$ and Λ_U are pairwise disjoint.

Set $x := |R|$, so $|S| = |T| = k - x$. Observe that, since $n \geq 2k$ we must have $|U| \geq x$. We split into two cases and we will show that our assumptions to this point lead to a contradiction.

1. SUPPOSE THAT WE CAN CHOOSE E_1, E_2 SO THAT $1 < x$. This means, in particular that both R and U have cardinality at least 2; hence Λ_R and Λ_U are all non-empty.

Let $E_3 \in \Lambda_R$. By counting we must have

$$E_3 = (E_3 \cap R) \cup U \quad \text{or} \quad (E_3 \cap R) \cup S \cup T.$$

Let $E_4 \in \Lambda_U$. By counting we must have

$$E_4 = (E_4 \cap U) \cup R \quad \text{or} \quad (E_4 \cap U) \cup S \quad \text{or} \quad (E_4 \cap U) \cup T \quad \text{or} \quad (E_4 \cap U) \cup S \cup T.$$

We will go through the various combinations and show that, in every case, the set $\{E_1, E_2, E_3, E_4\}$ is not independent, thereby giving our contradiction. In what follows $g \in G_{E_1, E_3, E_4}$.

Consider, first, the possibilities for E_3 .

(E3A) Suppose that $E_3 = (E_3 \cap R) \cup U$. Then g stabilizes $(E_1 \cup E_3)^C = T$ and $E_3 \setminus (E_1 \cap E_3) = U$.

(E3B) Suppose that $E_3 = (E_3 \cap R) \cup S \cup T$. Then g stabilizes $E_3 \setminus (E_1 \cap E_3) = T$ and $(E_1 \cup E_3)^C = U$.

Thus, in all cases, g stabilizes both T and U . Now consider the possibilities for E_4 .

(E4A) Suppose that $E_4 = (E_4 \cap U) \cup R$. Then g stabilizes $E_1 \cap E_4 = R$ and hence also $R \cup T = E_2$. This contradicts independence (the pointwise stabilizer of $\{E_1, E_3, E_4\}$ is equal to the pointwise stabilizer of $\{E_1, E_2, E_3, E_4\}$).

(E4B) Suppose that $E_4 = (E_4 \cap U) \cup S$. Then g stabilizes $E_1 \cap E_4 = S$, hence also $\Omega \setminus (S \cup T \cup U) = R$, hence also $R \cup T = E_2$. We have the same contradiction.

(E4C) Suppose that $E_4 = (E_4 \cap U) \cup S \cup T$. Then g stabilizes $E_1 \cap E_4 = S$, hence also $\Omega \setminus (S \cup T \cup U) = R$, hence also $R \cup T = E_2$. We have the same contradiction.

(E4D) Suppose that $E_4 = (E_4 \cap U) \cup T$. For this final case we swap g with an element $h \in G_{E_2, E_3, E_4}$, we swap S with T and we swap E_1 with E_2 . Now, with these changes, the arguments for (E3A) and (E3B) tell us that h stabilizes both S and U . Next the argument for (E4B) tells us that h stabilizes R , hence also $R \cup S = E_1$. Now we again have a contradiction (the pointwise stabilizer of $\{E_2, E_3, E_4\}$ is equal to the pointwise stabilizer of $\{E_1, E_2, E_3, E_4\}$).

2. SUPPOSE THAT $|E_1 \cap E_2| \in \{0, 1\}$ FOR ALL DISTINCT $E_1, E_2 \in \Lambda$. This is the remaining case. We fix an incident pair E_1 and E_2 and observe that Λ_S and Λ_T are non-empty. Let $E_3 \in \Lambda_S, E_4 \in \Lambda_T$; observe that $E_3 = (E_3 \cap S) \cup U$ and $E_4 = (E_4 \cap T) \cup U$. But then $|U| \leq |E_3 \cap E_4| \leq 1$, hence $|U| = 1$. This implies that

$$|E_3| = |E_3 \cap S| + |U| = |E_3 \cap E_1| + |U| = 1 + 1 = 2.$$

Thus $k = 2$, a contradiction. \square

Proof of Theorem 1.1 (2) for $k \geq 3$. The work in §2.4 implies that we need only prove an upper bound for $H(S_n, \Omega_k)$. Lemma 2.5 (2) yields the result if $\ell \geq 3$. Lemma 2.8 yields the result if $\ell = 1$. Assume, then, that $\ell = 2$. Lemma 2.7 yields the result if each component contains at least 2 edges. Lemma 2.5 (1) yields the result if there is a component with 1 edge. Lemma 2.8 yields the result if exactly one of the components has 0 edges. Finally if both components have 0 edges, then the fact that $n \geq 2k$ implies the result. \square

3. THE ALTERNATING GROUP

One naturally wonders to what extent the results given here extend to the action of A_n on k -sets. Throughout this section k and n will be positive integers with $k \leq \frac{n}{2}$.

For irredundant bases we can adjust the proof given in §2.1, making use of the following easy fact: suppose that \mathcal{P}_i and \mathcal{P}_j are partitions corresponding to a set of k -subsets in $\{1, \dots, n\}$ as described at the start of §2. Let H_i (resp. H_j) be the stabilizer in A_n of all parts of \mathcal{P}_i (resp. \mathcal{P}_j). If $H_i = H_j$, then either $\mathcal{P}_i = \mathcal{P}_j$ or else the two partitions are of type 1^n or $1^{n-2}2^1$.

Let us show how this observation yields the required result.

{p: I An}

Proposition 3.1.

$$I(A_n, \Omega_k) = \begin{cases} n - 2, & \text{if } \gcd(n, k) = 1; \\ \max(2, n - 3), & \text{otherwise.} \end{cases}$$

Proof. Let $G = S_n$ and suppose that

$$G \supseteq G_{\omega_1} \supseteq G_{\omega_1, \omega_2} \supseteq G_{\omega_1, \omega_2, \omega_3} \supseteq \dots \supseteq G_{\omega_1, \omega_2, \dots, \omega_e}$$

is a stabilizer chain corresponding to an irredundant base $[\omega_1, \dots, \omega_e]$. The observation implies that we have

$$A_n \supseteq G_{\omega_1} \cap A_n \supseteq G_{\omega_1, \omega_2} \cap A_n \supseteq G_{\omega_1, \omega_2, \omega_3} \cap A_n \supseteq \dots \supseteq G_{\omega_1, \omega_2, \dots, \omega_{e-1}} \cap A_n \supseteq G_{\omega_1, \omega_2, \dots, \omega_e} \cap A_n$$

and hence either $[\omega_1, \dots, \omega_e]$ or $[\omega_1, \dots, \omega_{e-1}]$ is an irredundant base for A_n . This implies that $I(A_n, \Omega_k) = I(S_n, \Omega_k) - 1$ and Theorem 1.1 implies that

$$I(A_n, \Omega_k) \geq \begin{cases} n - 2, & \text{if } \gcd(n, k) = 1; \\ n - 3, & \text{otherwise.} \end{cases}$$

Now we will give an upper bound for $I(A_n, \Omega_k)$. Let $[\omega_1, \dots, \omega_e]$ be an irredundant base for the action of A_n on Ω_k . Then the observation above implies that $\mathcal{P}_{\{\omega_1, \dots, \omega_{e-1}\}}$ contains at most $n - 2$ parts. Applying Lemma 2.1 with $\Lambda = \{\omega_1, \dots, \omega_{e-1}\}$ and $\Delta = \emptyset$ implies that $e - 1 = |\Lambda| \leq n - 3$ and so $e \leq n - 2$. This yields the result when $\gcd(n, k) = 1$.

Suppose now that $\gcd(n, k) = g > 1$ and that $[\omega_1, \dots, \omega_{n-2}]$ is an irredundant base for the action of A_n on Ω_k ; we must show that then $n = 4$. The observation above implies that $\mathcal{P}_{\{\omega_1, \dots, \omega_{i+1}\}}$ has exactly one more part than $\mathcal{P}_{\{\omega_1, \dots, \omega_i\}}$ for $i = 1, \dots, n - 4$ and $\mathcal{P}_{\{\omega_1, \dots, \omega_{n-2}\}}$ has exactly two more parts than $\mathcal{P}_{\{\omega_1, \dots, \omega_{n-3}\}}$. Lemma 2.2 implies that all parts of $\mathcal{P}_{\{\omega_1, \dots, \omega_{n-3}\}}$ are divisible by g . But the type of $\mathcal{P}_{\{\omega_1, \dots, \omega_{n-3}\}}$ is either $2^{2^1}1^{n-4}$ or $3^{1^1}1^{n-3}$ and we conclude that $(n, k) = (4, 2)$ as required. The proof is completed by observing that $I(A_4, \Omega_2) = 2$. \square

For the other statistics in question it is easy to pin the value down within an error of 1; the next result does this.

{p: d}

Proposition 3.2. *Suppose that k and n are positive integers with $k \leq \frac{n}{2}$. Then*

$$(3.1) \quad \{\mathbf{e}: \mathbf{4}\} \quad H(S_n, \Omega_k) - 1 \leq B(A_n, \Omega_k) \leq H(A_n, \Omega_k) \leq H(S_n, \Omega_k).$$

Proof. The first inequality is obtained by observing that if we excise the final set from (2.1) and (2.2), then we obtain a minimal base for A_n .

The second inequality is elementary; it was given in (1.1). The third inequality, likewise, is an easy consequence of the definition of height. \square

All that remains, then, is to establish which of the two possible values holds for each value of k and n . Let us consider the situation for small values of k :

($k = 1$) It is immediate that $B(A_n, \Omega_1) = H(A_n, \Omega_1) = n - 2$, the smaller of the two possible values.

($k = 2$) We claim that in this case

$$B(A_n, \Omega_2) = H(A_n, \Omega_2) = \begin{cases} n - 3, & \text{if } n \neq 4; \\ 2, & \text{if } n = 4. \end{cases}$$

Thus, provided $n \neq 4$, we again obtain the smaller of the two possible values. To justify our claim we note first that the value for $n = 4$ is easy to obtain. When $n > 4$ it is sufficient to prove that $H(A_n, \Omega_2) \leq n - 3$. To see this we let Λ be an independent set and we form the graph Γ_Λ as in §2.3. Lemma 2.3 implies that, since Λ is independent, the graph Γ_Λ is a forest. If this forest has 3 or more connected components, then the result follows immediately, so we suppose that there are at most 2 components. If one of these components consists of a single edge, then deleting this edge results in a set of 2-sets whose pointwise stabilizer in A_n is trivial; this is a contradiction of the fact that Λ is independent. If all components contain 0 edges or at least 2 edges, then it is easy to check that either $n = 4$ or else deleting a leaf edge results in a set of 2-sets whose pointwise stabilizer in A_n is trivial. Again this is a contradiction and we are done.

($k = 3$) We claim that in this case $B(A_n, \Omega_3) = H(A_n, \Omega_3) = n - 3$ which is the larger of the two possible values, except when $n = 8$. To justify our claim we note first that the value for $n = 8$ is easy to obtain. When $n \neq 8$ it is sufficient to prove that $B(A_n, \Omega_3) \geq n - 3$. This follows simply by observing that the following set is a minimum base of size $n - 3$:

$$\left\{ \{1, 2, 3\}, \{1, 2, 4\}, \dots, \{1, 2, n - 1\} \right\}.$$

We have not investigated the case $k \geq 4$.

Finally, referring to §1.2, we remark that in [Che16], Cherlin calculated $\text{RC}(A_n, \Omega_k)$ precisely (correcting an earlier calculation in [Che00]). The comments above imply that for all k and n with $k \leq \frac{n}{2}$ we have

$$H(A_n, \Omega_k) \leq \text{RC}(A_n, \Omega_k) \leq H(A_n, \Omega_k) + 1.$$

Thus, unlike S_n , the relational complexity of the action of A_n on k -sets does indeed track height.

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