Strictly Concentric Magic Squares

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Abstract

Magic Squares as a mathematical structure have existed for 5000 years, yet they are still an interesting topic of new research. This thesis presents general definitions, examples and important properties of Strictly Concentric Magic Squares (SCMS). Using the known minimum centre cell value of Prime Strictly Concentric Magic Squares (PSCMS) of order 5, some structural properties are established, enabling the production of an algorithm for construction of minimum PSCMS. The number of minimum PSCMS of order 5 is enumerated.

Partial SCMS are then introduced with important definitions on completability of grids, with relation to known concepts in Latin Squares and Sudoku grids. The cardinality of sets for different types of completability are given in general, where possible, for grids of order n. The idea of unavoidable sets is introduced on SCMS before specific patterns for the minimum PSCMS of order 5 are given.

Having focused on PSCMS of order 5, this thesis then investigates the structure in general for PSCMS of higher odd order. Using the known minimum centre cell value of PSCMS of order 7, an algorithm for construction of these grids is given and the number of minimum PSCMS of order 7 is enumerated. PSCMS of even order are discussed briefly with definitions that differ from the odd order given as well as an algorithm for construction of a PSCMS of order 6.

The concept of water retention is introduced, firstly on Normal Number Squares, then Prime Number Squares before applying the concept to the minimum PSCMS of order 5. Definitions of patterns are given formally as well as a comparison of known results. Maximum water retention is found in specific cases and compared on identified types of minimum PSCMS of order 5.

Finally, this thesis concludes with a discussion of possible future work.

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Nomenclature

$(\bar{i}, \bar{j})$ The paired cell of (i, j).

(i, j) The cell of a grid in row i, column j.

$\mathbb{A}_n^{\mathbf{X}}$ The set of all SCMS to which the partial grid $\mathbf{X}_n$ can be completed.

$\mathbb{B}^n$ The set of all border pairs in every border of a SCMS of order n.

$\mathbb{B}_n$ The set of border pairs in the outer border of a SCMS of order n.

$\mathbb{N}$ The set of natural numbers.

$\mathbb{N}'$ Any subset of $\mathbb{N}$.

$\mathbb{P}$ The set of prime numbers.

$\mathbb{P}'$ Any subset of $\mathbb{P}$.

$\mathbb{V}_n^{\mathbf{A}}$ The set of all critical sets of $\mathbf{A}_n$.

$\mathbf{A_n}$ A grid with values in cells, represented as triples (i, j, a_{ij}).

$\mathbf{F}_n^{min}$ A minimal forced completable set of a SCMS of order n, a set of triples.

$\mathbf{F}_n$ A forced completable set of a SCMS of order n, a set of triples.

$\mathbf{H}_n$ A set of triples (i, j, a_{ij}) of the non-empty cells of a partial SCMS of order n.

$\mathbf{V}_n^{min}$ A minimal critical set of a SCMS of order n, a set of triples.

$\mathbf{V}_n$	A critical set of a SCMS of order n, a set of triples.
$\mathfrak{P}$	The set of permutation operations that SCMS can undergo.
A_n	A grid of cell positions (tuples (i, j)).
a_{ij}	A value in cell (i, j) of a grid.
B^n	The set of all border cells in every border of a SCMS of order n.
B_n	The set of border cells in the outer border of a SCMS of order n.
$D_n^{\mathbf{H},a}$	An unavoidable set of Form a on a partial SCMS, $\mathbf{H}_n$, (tuples (i, j)).
D_n	A set of cells that form an unavoidable set of a SCMS of order n (tuples (i, j)).
F_n	A set of cells that form a forced completable set of a SCMS of order n (tuples (i, j)).
H'_n	A set of tuples (i, j) of the empty cells of a partial grid H_n.
H_n	A set of tuples (i, j) of the non-empty cells of a partial grid of order n.
i	Row index of a grid.
j	Column index of a grid.
L	A Latin Square.
M	The centre cell value of a SCMS of order n, n odd, and half the sum of the paired values for a SCMS of order n, n even.
m	The order of a subsquare of a grid.
n	The order of a grid.
S_n	The magic constant of a SCMS of order n.
V_n	A set of cells that form a critical set of a SCMS of order n (tuples (i, j)).

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Candidate declaration

This is to certify that, except where specific reference is made, the work described in this thesis is the result of my own research. Neither this thesis, nor any part of it, has been presented, or is currently submitted, in candidature for any other award at this or any other University.

Signed ……………………………………….…
Candidate

Date …………………………………….......

Chapter 1

Literature Review and Notation

This thesis addresses Strictly Concentric Magic Squares with particular emphasis on squares of odd order that contain primes. While there has been interest in the literature on Magic Squares, their properties and enumeration, there is very limited material addressing Prime Concentric Magic Squares. Examples have been provided by hobbyists but there is an absence of any formal theoretical framework or mathematical underpinning or results in peer reviewed journals or archivable form. Hence, in providing necessary background material and definitions, new results and mathematical foundations for such squares are provided in this introductory chapter. These are highlighted where appropriate.

A literature review relating to Magic Squares and prime numbers is given in this chapter. Likewise, useful background material is provided on Latin Squares, which have similar properties to those of Magic Squares. Those definitions for Magic Squares that apply in general are given first, before presenting definitions that apply just to squares of odd order.

1.1 Magic Squares

Definition 1.1. *A* ***Magic Square*** *of order n is an n by n grid containing n^2 distinct integers positioned such that all rows, columns and main diagonals sum to the same value, S_n, known as the magic constant [1]. A* ***Normal Magic Square****, NMS, contains the integers 1 to n^2.*

*A **Prime Magic Square**, PMS, contains n^2 distinct primes.*

A Magic Square of order 2 does not exist as it is not possible to locate distinct integers into the four cells so that each row and column sums to the same magic constant. Consider a row containing the values a and p and a column containing the values a and q, where $a \neq p \neq q$, then $a + p \neq a + q$.

The position of the cell of a grid of order n at row i, $i = 1, \ldots, n$, and column j, $j = 1, \ldots, n$, is denoted (i, j) and has value a_{ij}. For example the centre cell of such a grid for n odd is in position $(\frac{n+1}{2}, \frac{n+1}{2})$ and has value $a_{\frac{n+1}{2}\frac{n+1}{2}}$.

Definition 1.2. *A **subsquare** of order m of a Magic Square of order n is comprised of the centre m by m cells of the Magic Square.*

Definition 1.3. *A **magic subsquare** of order m of a Magic Square of order n is a subsquare which is itself a Magic Square.*

Definition 1.4. *A Magic Square of order n for which its order $(n-2)$ subsquare is a magic subsquare is termed a **Concentric Magic Square**, CMS [2]. A CMS containing n^2 distinct primes is denoted a **Prime Concentric Magic Square**, PCMS.*

Definition 1.5. *If all sums of values in pairs of cells symmetric about the centre are equal then the Magic Square is referred to as **associative** [31].*

The above definitions are used to provide further definitions and properties for Magic Squares of odd order here and of even order in Chapter 4.

1.1.1 Magic Squares of Odd Order

The following definitions and results are provided by the current author to provide foundations for future results.

Lemma 1.6. *For n odd, the subsquares of a Magic Square of order n are of order $m = n - 2i$, $i = 1, \ldots, \frac{n-1}{2}$. The smallest such subsquare is of order 1.*

Proof. Proof follows immediately from Definition 1.2. □

The subsquare of order 1 is here considered a trivial magic subsquare, hence the smallest possible Concentric Magic Square is of order 3, which is itself considered trivial.

Definition 1.7. *A Magic Square of order n, $n \geq 5$ and odd, is* ***Strictly Concentric****, denoted a SCMS, if each of its order $m = n - 2i$ subsquares, $i = 1, \ldots, \frac{n-3}{2}$, is a CMS. A subsquare of order 3 is here considered a trivial SCMS. A SCMS containing n^2 distinct primes is denoted a PSCMS.*

In the construction and enumeration of SCMS later in this thesis, the concept of paired cells will be of importance due to constraints on the values in such cells.

Definition 1.8. *For a SCMS of order n, n odd, a cell in row i, column j has a* ***paired cell*** *in row $\bar{i}$, column $\bar{j}$, such that*

$$(\bar{i}, \bar{j}) = \begin{cases} (n-i+1, n-j+1) & i = 1, \ldots, n, \ i \neq \frac{n+1}{2}, \ j = i & (1) \\ (n-i+1, i) & i = 1, \ldots, n, \ i \neq \frac{n+1}{2}, \ j = n-i+1 & (2) \\ (i, n-j+1) & i = 2, \ldots, n-1, \ j \neq i, \quad i+j \leq n \quad \textit{when} \quad i > j \\ & \textit{and} \quad i+j \geq n+2 \quad \textit{when} \quad i < j & (3) \\ (n-i+1, j) & j = 2, \ldots, n-1, \ j \neq i, \quad i+j \leq n \quad \textit{when} \quad j > i \\ & \textit{and} \quad i+j \geq n+2 \quad \textit{when} \quad j < i & (4) \end{cases}$$

Figure 1.1 illustrates the conditions on paired cells given in Definition 1.8 for a SCMS of order 5.

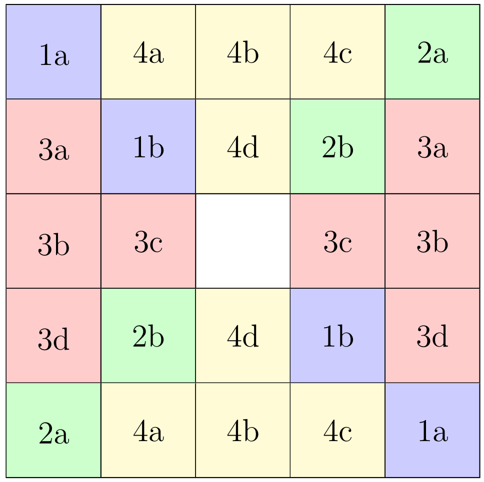


Figure 1.1: *Illustration of Paired Cells for a SCMS of Order 5; the Cell Numbers Relate to the Equation Numbers Given in Definition 1.8, Followed by a Letter Denoting Pairings*

Definition 1.9. *A SCMS of order n, n odd, and each of its subsquares, has a border which comprises those cells which are adjacent to its respective outer edge. Let B_n be the set of* ***border cells*** *of the SCMS of order n, and B_{n-2i} be the set of border cells of its subsquares of order $n-2i$, $i = 1, \ldots, \frac{n-3}{2}$. B^n denotes the set of all such border cells for an SCMS of order n; $B^n = \cup B_{n-2i}$, $i = 0, \ldots, \frac{n-3}{2}$.*

Definition 1.10. *A* ***border pair*** *$(a_{ij}, a_{\bar{i}\bar{j}})$ is a pair of values placed in cells in B^n for a SCMS of order n, where (i, j) and $(\bar{i}, \bar{j})$ are paired cells. Let $\mathbb{B}_n$ be the set of border pairs of the SCMS of order n, and $\mathbb{B}_{n-2i}$ be the set of border pairs of its subsquares of order $n-2i$, $i = 1, \ldots, \frac{n-3}{2}$. Let $\mathbb{B}^n$ denote the set of all such border pairs for a SCMS of order n; $\mathbb{B}^n = \cup \mathbb{B}_{n-2i}$, $i = 0, \ldots, \frac{n-3}{2}$, $|\mathbb{B}^n| = \frac{|B^n|}{2}$.*

Lemma 1.11. *The number of border pairs, $|\mathbb{B}^n|$, of a SCMS of order n, n odd, $n \geq 3$ is $|\mathbb{B}^n| = 2(n-1) + |\mathbb{B}^{n-2}|$ where $|\mathbb{B}^1|$ is taken to be 0.*

Proof. Let $n = 3$, $|\mathbb{B}^3| = 4$ from observation and satisfies the given recurrence. Assume the recurrence is true for some $n = k$, $k > 3$ and odd, $|\mathbb{B}^k| = 2(k-1) + |\mathbb{B}^{k-2}|$. Now consider the case $n = k+2$, for which $2((k+2)-1)$ border pairs are added to $\mathbb{B}^k$, hence $|\mathbb{B}^{k+2}| = 2((k+2)-1) + |\mathbb{B}^k|$. By induction the recurrence holds for any n odd. □

From the recurrence the integer sequence obtained is A046092 [36], and is discussed by the current author in [34].

Theorem 1.12. *The number of border pairs, $|\mathbb{B}^n|$, of a SCMS of order n, n odd, $n \geq 3$ is* $|\mathbb{B}^n| = \dfrac{n^2-1}{2}$.

Proof. For j odd, $j = (2i+1)$ where $i = 1, \ldots, \frac{n-1}{2}$, the number of pairs for each border of an order j subsquare is $|\mathbb{B}_j| = 2(j-1)$. From the proof of Lemma 1.11, the order n SCMS has $|\mathbb{B}_n| = 2(n-1)$ border pairs. Hence, $|\mathbb{B}_j| = 2((2i+1)-1) = 4i$. Hence, $|\mathbb{B}^n| = 4\sum_{i=1}^{\frac{n-1}{2}} i$. Since $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, then $4\sum_{i=1}^{\frac{n-1}{2}} i = \frac{4}{2}(\frac{n-1}{2})(\frac{n-1}{2}+1) = (\frac{n-1}{2})(n+1) = \frac{n^2-1}{2}$.
$|\mathbb{B}^n| = \dfrac{n^2-1}{2}$. □

Definition 1.13. *Denote the centre cell value of a SCMS of order n, n odd, by M.*

Lemma 1.14. *A Magic Square of order 3 has magic constant $3M$ and has four border pairs that each sum to $2M$.*

Proof. As the values in each column sum to the magic constant, S_3, then $a_{11} + a_{21} + a_{31} + a_{13} + a_{23} + a_{33} = 2S_3$. Consider the pairs a_{11}, a_{33} and a_{13}, a_{31} which form diagonals with a_{22}, and likewise the pair a_{21}, a_{23} which forms the centre row with a_{22}; these all sum to S_3.
As $M = a_{22}$, then $a_{11} + a_{33} = a_{13} + a_{31} = a_{21} + a_{23} = S_3 - M$
Hence, $2S_3 = 3(S_3 - M)$ and $S_3 = 3M$, and it follows that each border pair sums to $2M$. □

Lemma 1.15. *All Magic Squares of order 3 are associative.*

Proof. From Lemma 1.14, all values in pairs of cells symmetric about the centre sum to the same value ($2M$), hence all Magic Squares of order 3 are associative. □

It is shown in Chapter 2 that Concentric Magic Squares of order n, n odd and $n > 3$, cannot be associative.

Example 1.16. *A PSCMS of order $n = 7$ is given in Figure 1.2 [24]. The shading (dark grey, light grey and white) is employed to highlight that the PCMS is strictly concentric, specifically that it has a magic subsquare of order $n - 2 = 5$ which is itself a PSCMS and a magic subsquare of order $n - 4 = 3$ which is a trivial PSCMS.*

6367	4597	4723	6577	4513	4831	6451
4603	5527	4993	5641	6073	4951	6271
4663	4657	9007	1861	5443	6217	6211
6547	5227	1873	5437	9001	5647	4327
4783	5851	5431	9013	1867	5023	6091
6673	5923	5881	5233	4801	5347	4201
4423	6277	6151	4297	6361	6043	4507

Figure 1.2: *A Prime Strictly Concentric Magic Square of Order 7 [24]*

W.S. Andrews has published extensively on Magic Squares and has specifically explored Prime Magic Squares having the lowest possible magic constant [3]. In this thesis it is formalised that for Strictly Concentric Magic Squares of odd order the lowest possible magic constant is dependent on the centre cell value, M.

Definition 1.17. *A SCMS of order n that has minimum M is termed a* ***minimum SCMS****. When the values in the SCMS are primes this is termed a* ***minimum PSCMS****.*

1.2 Prime Numbers

A prime number is a positive integer which is divisible only by itself and 1, and by Euclid's Theorem there are known to be infinitely many such primes [43]. The focus of study for this thesis is the PSCMS and therefore it is necessary to note which primes can be included in such grids.

Definition 1.18. *The difference between successive primes, p_k and p_{k+1} is $G_k = p_{k+1} - p_k$. G_k is also referred to as the **prime gap** following the prime in the kth position p_k and the next possible prime p_{k+1} [27].*

The topic of prime distribution, and in particular the finding of patterns of prime gaps, has been the focus of much research. Just under 45% of prime gaps, up to the 3×10^7th prime number, are of the form $6k$, $k \in \mathbb{N}$ [38]. In this thesis the gaps considered are between relatively small primes. In Chapters 2 and 4 all the primes used in the PSCMS of order 5 and the PSCMS of order 7 have gaps of the form $6k$.

In Chapter 2 it is proved that a PSCMS of order n, n odd, can only contain prime numbers of one of the forms $6k+1$ or $6k-1$, $k \in \mathbb{N}$ and hence a PSCMS of order n, n odd, can never include the integer 3, since 3 is neither of the form $6k+1$ nor $6k-1$, $k \in \mathbb{N}$.

1.3 Latin Squares

Latin Squares are structurally similar to Magic Squares. The same types of permutation operations can be applied to both types of grid and similar enumerative techniques, using permutations, are suitable for them both. Other structural properties and known concepts identified in Latin Squares are also relevant in Chapter 3, where they are explored in relation to Magic Squares.

Definition 1.19. *A **Latin Square** of order n is a square grid with n^2 entries from n different elements, with no element occurring twice within any row or column of the grid [17]. Typically these use the symbol set 1, 2, ..., n or 0, 1, ..., $n-1$.*

Definition 1.20. *A Latin Square of order n (on the symbol set 1, 2, ..., n or 0, 1, ..., $n-1$) is referred to as **reduced**, or in **standard form**, if in the first row and the first column the elements occur in natural order [8].*

Figure 1.3 gives an example of a reduced Latin Square of order 3.

1	2	3
2	3	1
3	1	2

Figure 1.3: *A Reduced Latin Square of Order 3*

Definition 1.21. *Two Latin Squares L and L' of order n are* ***isomorphic*** *if there is a bijection $\phi : S \to S$ such that $\phi L(i, j) = L'(\phi(i), \phi(j))$ for every i, j in S, where S is not only the symbol set of each square, but also the indexing set for the rows and columns of each square [8]. An isomorphism that maps L to itself is an automorphism.*

An example of a reduced Latin Square of order 9 is given in Figure 1.4(a) and an isomorphic Latin Square is given in Figure 1.4(b). A similar reduced form argument is used in Chapter 2 for all examples of PSCMS.

1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	1
3	4	5	6	7	8	9	1	2
4	5	6	7	8	9	1	2	3
5	6	7	8	9	1	2	3	4
6	7	8	9	1	2	3	4	5
7	8	9	1	2	3	4	5	6
8	9	1	2	3	4	5	6	7
9	1	2	3	4	5	6	7	8

(a) A Reduced Latin Square of order 9

1	2	3	4	5	6	8	7	9
2	3	4	5	6	7	9	8	1
5	6	7	8	9	1	3	2	4
3	4	5	6	7	8	1	9	2
4	5	6	7	8	9	2	1	3
6	7	8	9	1	2	4	3	5
7	8	9	1	2	3	5	4	6
8	9	1	2	3	4	6	5	7
9	1	2	3	4	5	7	6	8

(b) A Latin Square Isomorphic to (a)

Figure 1.4: *Two Isomorphic Latin Squares of Order 9*

The definition of a critical set in a Latin Square is taken directly from [9] and given here as a similar definition is used in Section 3.3 for critical sets of SCMS.

Definition 1.22. *A **critical set** in a Latin Square L, of order n, is a set* $A = \{(i, j, k) | i, j, k \in \{1, \ldots, n\}\}$ *such that:*

(1) L is the only Latin Square of order n which has entry i in position (j, k) for each $(i, j, k) \in A$*;*

(2) no proper subset of A satisfies (1).

1.4 Equivalent Magic Squares

Definition 1.23. *Two Magic Squares* $\mathbf{A}_n^1$ *and* $\mathbf{A}_n^2$*, with values* a_{ij} *and* b_{ij} *respectively, are **equal** if* $a_{ij} = b_{ij}$ *for all cells* (i, j)*.*

Two isomorphic Latin Squares can be referred to as equivalent as one can be mapped to the other. Similar to isomorphisms in Latin Squares, all SCMS can undergo permutations, that are here defined, to form other equivalent SCMS.

Definition 1.24. *Two Strictly Concentric Magic Squares* $\mathbf{A}_n^1$ *and* $\mathbf{A}_n^2$ *are **equivalent** if one can be obtained from the other by undergoing permutations of the rows, columns or border pairs in cells,* $\mathbb{B}^n$*, given in Table 1.1, while maintaining all Magic Square properties including the magic constant. Otherwise* $\mathbf{A}_n^1$ *and* $\mathbf{A}_n^2$ *are **non-equivalent**.*

Permutation	**Order**
Permutation of border pairs in columns 2, ..., $n-1$	$(n-2)!$
Permutation of border pairs in rows 2, ..., $n-1$	$(n-2)!$
Permute row 1 and row n	2
Permute column 1 and column n	2
Interchange row 1 with column n and row n with column 1	2

Table 1.1: *Permutation Operations on the Border Pairs in Cells of a SCMS of Order n, Where the Permutation Order Given is the Smallest Number of Such Permutations Required to Return to the Initial State*

The cardinality of the set of operations, $\mathfrak{P}$, on the SCMS is given by the product of the orders of each permutation operation in Table 1.1:

$$|\mathfrak{P}| = 2^3 \times [(n-2)!]^2 \tag{1.1}$$

Every SCMS of order n, n odd, along with its subsquares of order $n-2i$, $i = 1, \ldots, \frac{n-3}{2}$, can undergo these permutations, and every permutation forms an equivalent SCMS.

Figure 1.5 shows two Magic Squares which are equivalent but not equal.

2	7	6
9	5	1
4	3	8

(a) A Magic Square of Order 3

2	9	4
7	5	3
6	1	8

(b) A Magic Square of Order 3 which is Equivalent But Not Equal to (a)

Figure 1.5: *Two Equivalent Magic Squares of Order 3*

1.5 Enumeration in the Literature

There is interest in the enumeration of Magic Squares in general, but thus far the number of Magic Squares is only known up to $n = 5$ for NMS. It is an unsolved problem to determine the number of Magic Squares of an arbitrary order. Frénicle de Bessy enumerated the 880 NMS of order 4 in 1693 [28] and Schroeppel enumerated the 275,305,224 NMS of order 5 in 1973 [23]; both of these enumerations were found by exhaustive search. Statistical tests and algorithms can be used for prediction, such as the Monte Carlo Method [30], to give an estimate of the number of higher order NMS and randomised algorithms can be used to enumerate using quasi-polynomials [5]. The number of NMS of order 6 has been estimated by Xin using a Euclid style algorithm in 2012 [44]. NMS with additional properties have also been enumerated, for example the number of associative Normal Magic Squares of order 7 were enumerated by Kato and Minato in 2019 [14]. To enumerate Magic Squares that do not have the normal property, simplifications have been made, such as neglecting the diagonal sums (i.e. treating Magic Squares as though they are structures termed **semi-magic** [40]) or neglecting the requirement that the integers within the Magic Squares are distinct [6],[37]. The above simplifications have been made due to larger order cases seeming computationally infeasible. In Chapter 2, the magic constant is fixed in order to enumerate PSCMS of order 5. It is intuitive to narrow down the enumeration in order to perform any analysis, as there are infinitely many primes and therefore it could be conjectured that there exist infinitely many PSCMS.

1.6 Structure of Thesis

In Chapter 2 some additional properties of PSCMS of odd order are presented, followed by a novel analysis of the minimum PSCMS of order 5. This includes a proof of the value of M for the minimum PSCMS of order 5 as well as providing details on how to construct the grid to satisfy the constraints of a Magic Square, and an enumeration of the minimum PSCMS of order 5.

Chapter 3 investigates the completability of partial SCMS using known concepts identified in both Latin Squares and Sudoku grids, namely critical sets and unavoidable sets. Unavoidable sets are then defined for minimum PSCMS of order 5.

PSCMS of higher order are the subject of Chapter 4. Firstly, the minimum PSCMS of order 7 is considered and an algorithm for construction and a full enumeration presented. This is followed by an algorithm for the construction for PSCMS of general odd order. PSCMS of even order are then introduced, with necessary general definitions given where they differ from the odd order definitions, before a minimum PSCMS of order 6 is discussed with an algorithm for construction. An algorithm is then presented for the construction for PSCMS of general even order.

In Chapter 5 the idea of water retention in a Magic Square is introduced. The literature review for this topic is given as an introduction to the chapter as the literature differs from the rest of the thesis. Some basic definitions are given for Number Squares before introducing water retention to Magic Squares. This concept is then applied to the minimum PSCMS of order 5 from Chapter 2.

Finally Chapter 6 is a conclusion chapter to summarise the main results in this thesis and identify some possible avenues for future work.

1.6.1 Significant Contributions of the Thesis

Chapter 2 provides the enumeration of a subclass of Magic Squares. The number of non-equivalent minimum PSCMS of order 5 is 35 and therefore there are 80,640 minimum PSCMS of order 5. In Chapter 3 the cardinality of the minimal forced completable set of a SCMS of order n, n odd is given as

$$|\mathbf{F}_n^{min}| = \frac{1}{2}(n^2 - 2n + 3).$$

A bound is given for the cardinality of a minimal critical set of a SCMS of order n, n odd, $n \geq 5$

$$\frac{1}{2}(n^2 - 4n + 9) \leq |\mathbf{V}_n^{min}| \leq \frac{1}{2}(n^2 - 2n + 3).$$

A complete classification of the unavoidable sets of minimum PSCMS of order 5 is given with the minimum cardinality of the different unavoidable sets calculated. No formal treatment of SCMS has previously been published, and hence these results along with the new definitions provide a framework useful for further work in this area.

Chapter 2

Prime Strictly Concentric Magic Squares of Order 5

2.1 Introduction

Using the definitions from Chapter 1, this chapter provides novel, foundational work on the properties of SCMS and addresses the absence in the literature of enumeration of SCMS; this chapter focuses on PSCMS of order 5. Firstly, some additional properties of PSCMS of odd order are given, and where the properties are general for SCMS they are given as such. The proof of the minimum PSCMS of order 5 is presented, followed by a construction and enumeration of minimum PSCMS of order 5. While work has addressed Concentric Magic Squares, most results presented concern Normal Magic Squares. Such work offers constructions for odd and even order [7][10], and notes primarily that methods of construction are somewhat complicated [33]. Prime Strictly Concentric Magic Squares have been addressed far less; one noteworthy PSCMS of order 13 formed by a hobbyist was published in [24] without construction. Makarova [26] observed the following relationship for which no proof is evident in the literature, and so is here provided by the current author.

Lemma 2.1. *A SCMS of order n, n odd, with centre cell value M has magic constant $S_n = nM$.*

Proof. By Lemma 1.14, $S_3 = 3M$. Assume that for some $n = k$, $k > 3$ and odd, $S_k = kM$. Now consider the case $n = k + 2$. Using arguments similar to Lemma 1.14, as the values in each column sum to the magic constant, S_{k+2}, take the first and the $(k+2)^{th}$ columns, then $a_{11} + a_{(k+2)(k+2)} + a_{(k+2)1} + a_{1(k+2)} + \sum_{i=2}^{k+1} a_{i1} + \sum_{i=2}^{k+1} a_{i(k+2)} = 2S_{k+2}$. Consider the pairs a_{11}, $a_{(k+2)(k+2)}$ and $a_{(k+2)1}$, $a_{1(k+2)}$ which form diagonals with cells in the subsquare of order k; these sum to S_{k+2}. Likewise the pairs a_{i1}, $a_{i(k+2)}$, $i = 2, \ldots, k+1$, which form the centre rows with the subsquare of order k; these also all sum to S_{k+2}. Since $S_k = kM$, then $a_{11} + a_{(k+2)(k+2)} = a_{(k+2)1} + a_{1(k+2)} = S_{k+2} - kM$ likewise $a_{i1} + a_{i(k+2)} = S_{k+2} - kM$, $i = 2, \ldots, k+1$. Hence, $2S_{k+2} = (k+2)(S_{k+2} - kM)$ and therefore $S_{k+2} = (k+2)M$. By induction the recurrence holds for any n odd. Hence, for all n odd, $S_n = nM$. □

Recall the definition of border pairs (Definition 1.10).

Lemma 2.2. *The elements of each border pair of a SCMS of order n, n odd, sum to $2M$ where M is the centre cell value of the SCMS.*

Proof. Consider a SCMS of order n, $n > 3$ and odd. By Lemma 2.1 the SCMS has magic constant $S_n = nM$. Removing the cells in B_n, and the corresponding border pair values in those cells, yields a magic subsquare of order $n-2$. The magic subsquare has magic constant $S_{n-2} = (n-2)M = nM - 2M$. Returning to the SCMS of order n, each row, column and main diagonal contains one border pair, in the outer border, and hence the elements of each border pair must sum to $2M$. By Lemma 1.14, when $n = 3$ the four border pairs in $\mathbb{B}_3$ sum to $2M$, therefore the lemma holds for all n odd. □

Definition 2.3. *Denote two values summing to $2M$ as a* ***pair of complement values****. Hence, all border pairs of a SCMS are pairs of complement values. Two prime numbers summing to $2M$ are therefore denoted as a* ***pair of complement primes****. Hence, all border pairs of a PSCMS are pairs of complement primes.*

Recall from Lemma 1.15 that all Magic Squares of order 3 are associative.

Lemma 2.4. *A Concentric Magic Square of order n, $n > 3$ and odd, cannot be associative.*

Proof. The positions of paired cells in a Strictly Concentric Magic Square of odd order are provided in Definition 1.8, and from Lemma 2.2 the values in these paired cells have the same sum. Assume for contradiction that the Magic Square is associative; the pairs symmetric about the centre sum to the same value. To fulfil both of these requirements there needs to be repeated values in cells. Therefore a Concentric Magic Square of order n, $n > 3$ and odd, cannot be associative. Only for $n = 3$ are all the positions of paired cells symmetric about the centre. □

Lemma 2.5. *A PSCMS of order n, n odd, can only contain prime numbers of one of the forms $6k + 1$ or $6k - 1$, $k \in \mathbb{N}$.*

Proof. It is well known that all prime numbers greater than 3 can be written in the form $6k + 1$ or $6k - 1$. By Lemma 2.2 each border pair of a PSCMS of order n, n odd, must sum to twice the centre cell value. In order for this property to hold, every entry must be of the same form as the centre cell value, which is itself of the form either $6k + 1$ or $6k - 1$. □

Corollary 2.6. *A PSCMS of order n, n odd, can never include the integer 3.*

Proof. By Lemma 2.5, since 3 is neither of the form $6k + 1$ nor $6k - 1$, $k \in \mathbb{N}$. □

Recall Section 1.4 on equivalent Magic Squares. Every SCMS of order n, n odd, along with its subsquares of order $n - 2i$, $i = 1, \ldots, \frac{n-3}{2}$, can undergo the permutations given in Table 1.1, and every permutation forms an equivalent SCMS. Figure 2.1 shows two PSCMS which are equivalent, in which the grid of order 5 in Figure 2.1(a) undergoes a permutation of the outer border pairs in rows 2, 3 and 4 to form the grid in Figure 2.1(b).

311	11	113	401	419
149	461	23	269	353
263	59	251	443	239
449	233	479	41	53
83	491	389	101	191

(a) A PSCMS

311	11	113	401	419
263	461	23	269	239
449	59	251	443	53
149	233	479	41	353
83	491	389	101	191

(b) A PSCMS Equivalent to (a)

Figure 2.1: *Two Equivalent PSCMS of Order 5*

2.2 Minimum PSCMS of Order 5

2.2.1 Introduction

A SCMS of order 5 comprises a centre cell value, M, and twelve distinct pairs of values summing to $2M$, four of which surround the centre cell forming the subsquare of order 3, and eight of which form the border of order 5. One such border example is shown in Figure 2.2(a) with one border pair shaded in grey. By Lemma 2.1, a SCMS of order 5 has magic constant $S_5 = 5M$.

Recall, from Definition 1.17, a minimum PSCMS has minimum M value. A minimum PSCMS of order 5 is given in [25] to have $S_5 = 1255$ with centre cell value $M = 251$, and border pairs summing to $2M = 502$. The magic constant for the subsquare of order 3 is $3M = 753$, as is evident in Figure 2.2(b). It is proved by the current author in Lemma 2.7 that the minimum PSCMS of order 5 has $M = 251$ and hence from Lemma 2.1, $S_5 = 1255$ and therefore the placement of primes in Figure 2.2 form a minimum PSCMS of order 5.

311	53	71	419	401
113				389
239		251		263
491				11
101	449	431	83	191

(a) Border of a Minimum PSCMS of Order 5 with Centre Cell Value 251 with one Border Pair Shaded in Grey

	461	23	269	
	59	251	443	
	233	479	41	

(b) Magic Subsquare of Order 3 for a Minimum PSCMS of Order 5

Figure 2.2: *Border and Magic Subsquare of Order 3 for a Minimum PSCMS of Order 5*

The following notation is used: denote by $\mathbb{P}$ the set of all prime numbers, and $\mathbb{P}'$ any subset of $\mathbb{P}$.

Lemma 2.7. *The minimum PSCMS of order 5 has centre cell value of 251, and hence magic constant of 1255.*

Proof. Assume for contradiction that the centre cell value, M, is less than 251. Let $\mathbb{P}$ be the set of all prime numbers. To construct a PSCMS for which $M < 251$ and prime, there must exist at least twelve distinct pairs of complement primes $x_i, \bar{x}_i \in \mathbb{P}$ such that $x_i + \bar{x}_i = 2M$, where $x_i, \bar{x}_i$ are the values in paired cells. These pairs of complement primes form $\mathbb{P}'$. Four of these pairs form the border pairs of the subsquare of order 3, and eight form the outer border pairs of the PSCMS of order 5. Only for $M = 233$ are there as many as twelve distinct pairs of complement primes. However, one of these pairs contains the integer '3' which from Corollary 2.6 cannot appear in a PSCMS. It is known that a PSCMS of order 5 with centre cell value 251 exists, and one example is given in Figure 2.3. Hence, any PSCMS of order 5 with centre cell value 251 and magic constant of 1255 is a minimum PSCMS. $\square$

311	53	71	419	401
113	461	23	269	389
239	59	251	443	263
491	233	479	41	11
101	449	431	83	191

Figure 2.3: *A Minimum PSCMS of Order 5*

2.2.2 Construction of Minimum PSCMS of Order 5

Having established M for all minimum PSCMS of order 5, this section details the construction of such a square. Given the centre cell value $M = 251$, there are thirteen pairs of complement primes satisfying Lemma 2.2. With trial and error, it can easily be found that there are only two possible non-equivalent magic subsquares of order 3, shown in Figure 2.4.

461	23	269
59	251	443
233	479	41

(a) Magic Subsquare 1

431	83	239
59	251	443
263	419	71

(b) Magic Subsquare 2

Figure 2.4: *The Two Non-Equivalent Magic Subsquares of Order 3 for a Minimum PSCMS of Order 5*

All minimum PSCMS of order 5 consist of a centre cell value and twelve pairs of complement primes formed from the following set of 26 prime numbers: $\mathbb{P}' = \{11,23,41,53,59,71,83,101,113, 149,191,233,239,263,269,311,353,389,401,419,431,443,449,461,479,491\}$. For subsquare 1 (Figure 2.4(a)), six different combinations of the required eight pairs of complement primes, from $\mathbb{P}'$, can be used to form a minimum PSCMS. For subsquare 2 (Figure 2.4(b)), nine different combinations of the required eight pairs of complement primes, from $\mathbb{P}'$, can be used to form a minimum PSCMS. These different combinations were found by hand using exhaustive search and are each referred to as a **type** and are given in Table 2.1.

Type	**List of Primes in the Border of Order 5**
1A	11, 53, 71, 83, 101, 113, 191, 239, 263, 311, 389, 401, 419, 431, 449, 491
1B	11, 53, 83, 101, 113, 149, 191, 239, 263, 311, 353, 389, 401, 419, 449, 491
1C	11, 71, 83, 101, 113, 149, 191, 239, 263, 311, 353, 389, 401, 419, 431, 491
1D	53, 71, 83, 101, 113, 149, 191, 239, 263, 311, 353, 389, 401, 419, 431, 449
1E	11, 53, 71, 83, 101, 113, 149, 239, 263, 353, 389, 401, 419, 431, 449, 491
1F	11, 53, 71, 83, 101, 149, 191, 239, 263, 311, 353, 401, 419, 431, 449, 491
2A	11, 23, 41, 53, 101, 113, 149, 191, 311, 353, 389, 401, 449, 461, 479, 491
2B	11, 23, 41, 53, 101, 113, 149, 233, 269, 353, 389, 401, 449, 461, 479, 491
2C	11, 23, 41, 53, 101, 113, 191, 233, 269, 311, 389, 401, 449, 461, 479, 491
2D	11, 23, 41, 101, 113, 149, 191, 233, 269, 311, 353, 389, 401, 461, 479, 491
2E	11, 23, 41, 53, 101, 149, 191, 233, 269, 311, 353, 401, 449, 461, 479, 491
2F	11, 23, 41, 53, 113, 149, 191, 233, 269, 311, 353, 389, 449, 461, 479, 491
2G	11, 23, 53, 101, 113, 149, 191, 233, 269, 311, 353, 389, 401, 449, 479, 491
2H	11, 41, 53, 101, 113, 149, 191, 233, 269, 311, 353, 389, 401, 449, 461, 491
2I	23, 41, 53, 101, 113, 149, 191, 233, 269, 311, 353, 389, 401, 449, 461, 479

Table 2.1: *Primes Used in the Border of the Minimum PSCMS of Order 5, B_5, with Magic Subsquares 1 and 2*

In order to construct a PSCMS of order 5, first we must construct the magic subsquare of order 3. A PSCMS of order 3 for any given M (if one exists) can be generated using Algorithm 1.

Algorithm 1 *Algorithm to form a PSCMS of order 3*

Begin
Input M and form $\mathbb{P}'$, the set of all primes that form pairs summing to $2M$.
Place M into the centre of an empty grid of order 3.
repeat
    Take a set of three distinct non-paired primes from $\mathbb{P}'$ that sum to $3M$, to form a set S, and their complements to form a set $\bar{S}$.
    Take a prime from $\mathbb{P}'$ and call it T, and its complement $\bar{T}$.
    repeat
        Take an element x of S, and an element y of $\bar{S}$ that is not paired with x.
        Let X be the sum of x, y and T
    until $X = 3M$, or no further combinations of x, y are possible.
until $X = 3M$, or no further combinations of S are possible.
if $X = 3M$ **then**
    Begin
    Place x in $(1, 1)$, y in $(1, 3)$ removing them from S and $\bar{S}$, and place their complements in the paired cells, removing them from $\bar{S}$ and S.
    Place T in $(1, 2)$ and place $\bar{T}$ in $(3, 2)$
    Place the remaining element of S in $(2, 1)$ and place its complement from $\bar{S}$ in $(2, 3)$.
    End
else
    no PSCMS exists for the placed M.
end if
End

Algorithm 2 uses Algorithm 1 to form a subsquare of order 3, with chosen M and $\mathbb{P}'$, before using a similar process to construct a PSCMS of order 5 around the given subsquare, if one exists.

Algorithm 2 *Algorithm to form a PSCMS of order 5*

Begin
Input M and form $\mathbb{P}'$, the set of all primes that form pairs summing to $2M$.
repeat
  Construct a PSCMS of order 3 with chosen M and $\mathbb{P}'$ using Algorithm 1.
  Place the magic subsquare of order 3, with centre cell value M, into the centre of an empty grid of order 5.
  Form a set Q of the primes from $\mathbb{P}'$ not used in the subsquare.
  repeat
    Take a set of five distinct non-paired primes from Q that sum to $5M$, to form a set S, and their complements to form a set $\bar{S}$.
    Take a set of three distinct non-paired primes from Q to form a set T, and their complements to form a set $\bar{T}$.
    repeat
      Take an element x of S, and an element y of $\bar{S}$ that is not paired with x.
      Let X be the sum of x, y and the elements of T.
    until $X = 5M$, or no further combinations of x, y are possible.
  until $X = 5M$, or no further combinations of S are possible.
until $X = 5M$, or no further PSCMS of order 3 can be generated using Algorithm 1.
if $X = 5M$ **then**
  Begin
  Place x in $(1,1)$, y in $(1,5)$ removing them from S and $\bar{S}$, and place their complements in the paired cells, removing them from $\bar{S}$ and S.
  Place the elements of T in $(1,2)$, $(1,3)$, $(1,4)$ in any order, and place their complements from $\bar{T}$ in the paired cells.
  Place the remaining elements of S in $(2,1)$, $(3,1)$, $(4,1)$ in any order and place their complements from $\bar{S}$ in the paired cells.
  End
else
  No PSCMS exists for the placed centre subsquare.
end if
End

Theorem 2.8. *A minimum PSCMS of order 5 with centre cell value 251 is always formed using Algorithm 2 when magic subsquare 1 or 2 (shown in Figure 2.4) is placed in the centre of the grid.*

Proof. For $M = 251$, the set of complement primes (primes summing to 502) $\mathbb{P}' = \{11,23,41,53,59,71,83,101,113,149,191,233,239,263,269,311,353,389,401,419,431,443,449,461,479,491\}$. Recall from Definition 2.3 that all border pairs are pairs of complement primes. Eight of these primes must be used in the subsquare and removed from $\mathbb{P}'$ to form a set

Q. In the case of subsquare 1, $Q = \{11,53,71,83,101,113,149,191,239,263,311,353,389,401,419,431,449,491\}$, and in the case of subsquare 2, $Q = \{11,23,41,53,101,113,149,191,233,269,311,353,389,401,449,461,479,491\}$. In both cases $|Q| = 18$, and Q consists of nine pairs of complement primes, of which eight pairs are needed to form the border of order 5.

It can easily be seen that five distinct pairs of complement primes satisfy conditions (1) and (3) below. From the remaining primes in Q it is easy to check that there is always three more pairs of complement primes that can be chosen to satisfy conditions (2) and (4) below.

(1) $a_{11} + a_{12} + a_{13} + a_{14} + a_{15} = 1255$

(2) $a_{11} + a_{21} + a_{31} + a_{41} + a_{51} = 1255$

(3) $a_{51} + a_{52} + a_{53} + a_{54} + a_{55} = 1255$

(4) $a_{15} + a_{25} + a_{35} + a_{45} + a_{55} = 1255$.

The primes are placed in the manner specified in Algorithm 2 (with the subsquare of order 3 having been determined in line 4, through a call to Algorithm 1, the paired values being determined in lines 7 to 15, and the border of the PSCMS of order 5 then being filled by lines 18 to 20). A minimum PSCMS is thereby formed. □

2.2.3 Enumeration of Minimum PSCMS of Order 5

Algorithm 2 and Tables 1.1 and 2.1 are utilised to determine the number of minimum PSCMS of order 5. Each of the fifteen types (given in Table 2.1) is enumerated separately, firstly in Section 2.2.3.1 the six types with magic subsquare 1 (shown in Figure 2.4(a)), and in Section 2.2.3.2 the nine types with magic subsquare 2 (shown in Figure 2.4(b)).

Within each type, the primes listed in Table 2.1 are placed in pairs of complement sets. That is, the elements of each set sum to the required magic constant, and each element of each set is uniquely paired with a complement element in the other. A unique label is given to each such pairing. The possible pairs are shown in Tables 2.2, 2.3, 2.4 and 2.5.

2.2.3.1 Enumeration With Magic Subsquare 1

The squares are divided into types such that the types do not share the same list of prime numbers; therefore a square of one type is non-equivalent to a square of another type. Recall that there are thirteen pairs of complement primes, and each type omits a different pair. Within each type there are non-equivalent variants which use different pairs of complement sets. Using Equation 1.1 each variant can undergo 2,304 permutations, using Table 1.1.

Enumeration of Type 1A

Table 2.2 shows the possible pairs of complement sets used in the construction of a minimum PSCMS of type 1A. These pairs, as well as those used in the enumeration of all other types, were found by hand using exhaustive search. Using these pairs it is possible to construct all variants and then employ these to determine the number of PSCMS of each type.

Type	Labels	Pairs of Complement Sets, S and $\bar{S}$
1A	A1	{53, 71, 311, 401, 419}, { 83, 101, 191, 431, 449}
	A2	{11, 113, 263, 419, 449}, { 53, 83, 239, 389, 491}
	A3	{11, 191, 263, 389, 401}, {101, 113, 239, 311, 491}
	A4	{53, 71, 239, 401, 491}, { 11, 101, 263, 431, 449}
	A5	{11, 113, 311, 401, 419}, { 83, 101, 191, 389, 491}

Table 2.2: *Pairs of Complement Sets of Primes, S and $\bar{S}$, Assigned to B_5 of the Minimum PSCMS of Type 1A*

Figure 2.5 shows examples of placements of the complement pairs from Table 2.2 around the magic subsquare of order 3 in order to form a PSCMS of order 5. This approach is consistent through all types and hence no other such examples are given.

Example 2.9.

419	71	311	401	53
11				491
113				389
263				239
449	431	191	101	83

(a) Type 1A using A1 and A2

401	53	71	419	311
11				491
263				239
389				113
191	449	431	83	101

(b) Type 1A using A1 and A3

401	53	71	239	491
113				389
311				191
419				83
11	449	431	263	101

(c) Type 1A using A4 and A5

Figure 2.5: *Three Borders for a Type 1A Minimum PSCMS of Order 5*

Lemma 2.10. *There are three non-equivalent PSCMS of type 1A.*

Proof. Given the primes shown in Table 2.1 for type 1A, there are five pairs of complement sets, shown in Table 2.2, for magic subsquare 1 that satisfy the constraints (1) to (4) in the proof of Theorem 2.8; there are only three non-equivalent squares which are generated by Algorithm 2, shown in Figure 2.5. These squares are non-equivalent as the values in the corner cells are different, and hence the permutation operations in Table 1.1 cannot be applied. □

Enumeration of Type 1B

Table 2.3 shows the possible pairs of complement sets used in the construction of a minimum PSCMS of type 1B.

Type	**Labels**	**Pairs of Complement Sets, S and $\bar{S}$**
1B	B1	{53, 191, 239, 353, 419}, {83, 149, 263, 311, 449}
	B2	{11, 191, 263, 389, 401}, {101, 113, 239, 311, 491}
	B3	{83, 101, 191, 389, 491}, {11, 113, 311, 401, 419}
	B4	{53, 149, 263, 389, 401}, {101, 113, 239, 353, 449}
	B5	{53, 101, 191, 419, 491}, {11, 83, 311, 401, 449}
	B6	{113, 149, 263, 311, 419}, {83, 191, 239, 353, 389}

Table 2.3: *Pairs of Complement Sets of Primes, S and $\bar{S}$, Assigned to B_5 of the Minimum PSCMS of Type 1B*

Lemma 2.11. *There are five non-equivalent PSCMS of type 1B.*

Proof. Given the primes shown in Table 2.1 for type 1B, there are six pairs of complement sets, shown in Table 2.3, for magic subsquare 1 that satisfy the constraints (1) to (4) in the proof of Theorem 2.8; there are only five non-equivalent squares which are generated by Algorithm 2. These squares are non-equivalent as for each of the five, either the combination of paired cells in columns and rows 2, 3, 4 differ, or the values in the corner cells are different, and hence the permutation operations in Table 1.1 cannot be applied. □

Enumeration of Types 1C, 1D, 1E and 1F

Table 2.4 shows the possible pairs of complement sets used in the construction of a minimum PSCMS of types 1C, 1D, 1E and 1F.

Type	**Labels**	**Pairs of Complement Sets, S and $\bar{S}$**
1C	C1	{11, 71, 353, 401, 419}, {83, 101, 149, 431, 491}
	C2	{83, 191, 239, 353, 389}, {113, 149, 263, 311, 419}
	C3	{11, 113, 311, 401, 419}, {83, 101, 191, 389, 491}
	C4	{101, 149, 263, 311, 431}, {71, 191, 239, 353, 401}
	C5	{11, 149, 263, 401, 431}, {71, 101, 239, 353, 491}
	C6	{101, 113, 191, 419, 431}, {71, 83, 311, 389, 401}
1D	D1	{53, 113, 239, 419, 431}, {71, 83, 263, 389, 449}
	D2	{71, 101, 311, 353, 419}, {83, 149, 191, 401, 431}
	D3	{71, 191, 239, 353, 401}, {101, 149, 263, 311, 431}
1E	E1	{11, 71, 353, 401, 419}, {83, 101, 149, 431, 491}
	E2	{101, 113, 239, 353, 449}, {53, 149, 263, 389, 401}
	E3	{71, 83, 263, 389, 449}, {53, 113, 239, 419, 431}
	E4	{11, 149, 263, 401, 431}, {71, 101, 239, 353, 491}
1F	F1	{53, 101, 191, 419, 491}, {11, 83, 311, 401, 449}
	F2	{101, 149, 263, 311, 431}, {71, 191, 239, 353, 401}
	F3	{11, 101, 263, 431, 449}, {53, 71, 239, 401, 491}
	F4	{71, 101, 311, 353, 419}, {83, 149, 191, 401, 431}
	F5	{11, 149, 263, 401, 431}, {71, 101, 239, 353, 491}
	F6	{53, 71, 311, 401, 419}, {83, 101, 191, 431, 449}

Table 2.4: *Pairs of Complement Sets of Primes, S and $\bar{S}$, Assigned to B_5 of the Minimum PSCMS of Types 1C, 1D, 1E and 1F*

Lemma 2.12. *There are:*

(1) three non-equivalent PSCMS of type 1C;

(2) two non-equivalent PSCMS of type 1D;

(3) three non-equivalent PSCMS of type 1E;

(4) three non-equivalent PSCMS of type 1F.

Proof. Cases (1), (2) and (3) follow similarly to Lemma 2.10, with (1) having six pairs of complement sets and three non-equivalent squares, (2) having three pairs of complement sets and two non-equivalent squares, and (3) having four pairs of complement sets and three non-equivalent squares. Case (4) follows similarly to Lemma 2.11 with six pairs of complement sets and three non-equivalent squares. □

Lemma 2.13. *There are 19 non-equivalent minimum PSCMS of order 5 with magic subsquare 1.*

Proof. This proof follows directly from Lemmas 2.10, 2.11 and 2.12. □

Theorem 2.14. *There are 43,776 minimum PSCMS of order 5 with magic subsquare 1.*

Proof. This proof follows directly from Lemma 2.13 and Equation 1.1. □

2.2.3.2 Enumeration With Magic Subsquare 2

The squares are divided into non-equivalent types in the same manner as subsquare 1 in Section 2.2.3.1. Table 2.5 shows the possible pairs of complement sets used in the construction of a minimum PSCMS with magic subsquare 2.

Type	Labels	Pairs of Complement Sets, S and $\bar{S}$
2A	A1	{11, 41, 353, 401, 449}, {53, 101, 149, 461, 491}
	A2	{11, 113, 191, 461, 479}, {23, 41, 311, 389, 491}
	A3	{11, 101, 311, 353, 479}, {23, 149, 191, 401, 491}
	A4	{23, 41, 353, 389, 449}, {53, 113, 149, 461, 479}
2B	B1	{11, 101, 233, 449, 461}, {41, 53, 269, 401, 491}
	B2	{23, 41, 353, 389, 449}, {53, 113, 149, 461, 479}
	B3	{11, 113, 269, 401, 461}, {41, 101, 233, 389, 491}
	B4	{53, 101, 269, 353, 479}, {23, 149, 233, 401, 449}
	B5	{11, 149, 233, 401, 461}, {41, 101, 269, 353, 491}
	B6	{23, 113, 269, 401, 449}, {53, 101, 233, 389, 479}
2C	C1	{11, 53, 311, 401, 479}, {23, 101, 191, 449, 491}
	C2	{101, 113, 269, 311, 461}, {41, 191, 233, 389, 401}
	C3	{11, 113, 191, 461, 479}, {23, 41, 311, 389, 491}
	C4	{23, 113, 269, 401, 449}, {53, 101, 233, 389, 479}
2D	D1	{11, 101, 311, 353, 479}, {23, 149, 191, 401, 491}
	D2	{11, 113, 269, 401, 461}, {41, 101, 233, 389, 491}
	D3	{11, 113, 191, 461, 479}, {23, 41, 311, 389, 491}
	D4	{41, 191, 269, 353, 401}, {101, 149, 233, 311, 461}
2E	E1	{11, 53, 311, 401, 479}, {23, 101, 191, 449, 491}
	E2	{101, 149, 233, 311, 461}, {41, 191, 269, 353, 401}
2F	F1	{11, 113, 191, 461, 479}, {23, 41, 311, 389, 491}
	F2	{113, 149, 233, 311, 449}, {53, 191, 269, 353, 389}
2G	G1	{11, 53, 311, 401, 479}, {23, 101, 191, 449, 491}
	G2	{53, 191, 269, 353, 389}, {113, 149, 233, 311, 449}
	G3	{11, 101, 311, 353, 479}, {23, 149, 191, 401, 491}
2H	H1	{11, 41, 353, 401, 449}, {53, 101, 149, 461, 491}
	H2	{53, 191, 269, 353, 389}, {113, 149, 233, 311, 449}
	H3	{11, 113, 269, 401, 461}, {41, 101, 233, 389, 491}
2I	I1	{23, 113, 269, 401, 449}, {53, 101, 233, 389, 479}
	I2	{53, 149, 191, 401, 461}, {41, 101, 311, 353, 449}

Table 2.5: *Pairs of Complement Sets of Primes, S and $\bar{S}$, Assigned to B_5 of the Minimum PSCMS with Magic Subsquare 2*

Lemma 2.15. *There are:*

(1) two non-equivalent PSCMS of type 2A;

(2) three non-equivalent PSCMS of type 2B;

(3) two non-equivalent PSCMS of type 2C;

(4) two non-equivalent PSCMS of type 2D;

(5) one PSCMS of type 2E;

(6) one PSCMS of type 2F;

(7) two non-equivalent PSCMS of type 2G;

(8) two non-equivalent PSCMS of type 2H;

(9) one PSCMS of type 2I.

Proof. Cases (1), (3), (4), (7) and (8) follow similarly to Lemma 2.10, with (1), (3) and (4) each having four pairs of complement sets and two non-equivalent squares, and (7) and (8) each having three pairs of complement sets and two non-equivalent squares. Case (2) follows similarly to Lemma 2.11 with six pairs of complement sets and three non-equivalent squares. Cases (5), (6) and (9) have only two pairs of complement sets and therefore just one square. □

Lemma 2.16. *There are 16 non-equivalent minimum PSCMS of order 5 with magic subsquare 2.*

Proof. This proof follows directly from Lemma 2.15. □

Theorem 2.17. *There are 36,864 minimum PSCMS of order 5 with magic subsquare 2.*

Proof. This proof follows directly from Lemma 2.16 and Equation 1.1. □

2.2.4 Results

Lemma 2.18. *There are 35 non-equivalent minimum PSCMS of order 5.*

Proof. This proof follows directly from Lemmas 2.13 and 2.16. □

Theorem 2.19. *There are 80,640 minimum PSCMS of order 5, these have magic constant 1255.*

Proof. This proof follows directly from Theorems 2.14 and 2.17. □

2.2.5 Conclusion of Minimum PSCMS of Order 5

This chapter provides the first formal analysis of minimum PSCMS of order 5. It is proved that SCMS of order n, n odd, have magic constant $S_n = nM$. Other important properties are formally defined and novel algorithms are provided, the first to construct a magic subsquare of order 3 and the second to construct a PSCMS of order 5 around the given subsquare. The centre cell value of the minimum PSCMS of order 5 is $M = 251$. It is here established that there are two possible non-equivalent grids of order 3 that are valid subsquares. A full classification of types of minimum PSCMS of order 5 using the subsquares and the lists of primes is given, which facilitates the enumeration of 80,640 minimum PSCMS of order 5, 35 of which are non-equivalent. These foundations provide a framework for future study in enumeration, however for PSCMS of higher order, the value of M is larger, and hence there are more pairs of complement primes summing to $2M$. This enables a greatly increased number of valid magic subsquares of order 3, and hence the enumeration becomes more complex. If border cells are placed around any minimum PSCMS of order 5, then a grid of order 7 is formed. However, it is not possible to form a PSCMS of order 7 with the minimum PSCMS of order 5 as a subsquare.

Lemma 2.20. *A border cannot be placed around the minimum Prime Strictly Concentric Magic Square of order 5 to form a Prime Strictly Concentric Magic Square of order 7.*

Proof. Assume for contradiction that a border is placed around the minimum PSCMS of order 5. For a PSCMS of order 7 to be formed there must be twenty-four pairs of complement primes, x_i and $\bar{x}_i$ where $x_i \neq \bar{x}_i$ such that $x_i + \bar{x}_i = 502$. The PSCMS of order 5 requires twelve of these complement pairs and the other twelve are required in the outer border. There only exist thirteen distinct complement pairs that fulfil these requirements, hence a PSCMS of order 7 cannot be formed with the minimum PSCMS of order 5 as a subsquare. □

The minimum PSCMS of order 7 is detailed in Chapter 4 and the general concepts given in this chapter and Chapter 1 are utilised.

The minimum PSCMS of order 5 given in this chapter is now used in Chapter 3, to explore the concept of completability of partial grids and to analyse patterns of unavoidable sets. In Chapter 5 the maximum water retention is found on each type of minimum PSCMS of order 5.

Chapter 3

Partial Low Order SCMS and Completability

3.1 Introduction

The idea of incomplete grids is common to both Latin Squares and Sudoku literature, the latter being mainly for recreational puzzle solving. The same ideas are formally applied here to Magic Squares for both puzzle solving and for exploring the mathematical structure of specific grids. Keedwell [15] briefly considered partial Magic Squares along with Latin Squares and Sudoku grids, investigating critical sets and unique completability of Normal Magic Squares of orders 3 and 4. Keedwell postulated that Concentric Magic Squares may be the easiest of the Magic Squares to investigate for the determination of critical sets (defined for Latin Squares in Definition 1.22 and for SCMS in Definition 3.18) for grids of larger order. The terminology in this chapter follows from the literature on Latin Squares and is formally defined by the current author for SCMS [16].

The following novel results rely on the idea that a SCMS of order n, n odd, has a known centre cell value. This chapter focuses on SCMS of order n, n odd, but where the definitions apply to both odd and even orders they are given in general.

Definition 3.1. *A **triple** (i, j, a_{ij}) specifies a value a_{ij} in row i and column j of a grid.*

Definition 3.2. *Denote by A_n the cells of a grid of order n, i.e. a set of (i, j) tuples, and denote by $\mathbf{A}_n$ a set of triples (i, j, a_{ij}) where a_{ij} is the value in cell (i, j).*

Recall, from Definition 1.24, that two Magic Squares $\mathbf{A}_n^1$ and $\mathbf{A}_n^2$ are termed equivalent if one can be formed from the other using the permutation operations in Table 1.1 and non-equivalent otherwise. Denote the set of natural numbers by $\mathbb{N}$ and any subset of $\mathbb{N}$ by $\mathbb{N}'$; for a Normal Magic Square, $\mathbb{N}'$ is the set containing the first n^2 numbers. Recall, from Chapter 2, that $\mathbb{P}$ denotes the set of all prime numbers, and $\mathbb{P}'$ any subset of $\mathbb{P}$. There exist complement pairs of values $a_{ij}, \bar{a}_{ij} \in \mathbb{N}$ (or $\mathbb{P}$) such that $a_{ij} + \bar{a}_{ij} = 2M$, where M is the centre cell value of a SCMS (or PSCMS) of order n, n odd, and $a_{ij}, \bar{a}_{ij}$ are the values in paired cells.
Firstly, general SCMS are considered before discussing PSCMS. Recall, from Lemma 2.1, that the magic constant S_n of a SCMS of order n, n odd, is $S_n = nM$ where M is the centre cell value (Definition 1.13), and from Lemma 2.2 that all border pairs sum to $2M$. Hence, once the centre cell value of the SCMS is fixed, the pairs of values summing to $2M$ form the corresponding set $\mathbb{N}'$, likewise for a PSCMS the pairs of primes summing to $2M$ form the corresponding set $\mathbb{P}'$. As M increases, the number of combinations of values summing to $2M$ grows, and hence the cardinality of the set of values in $\mathbb{N}'$ (or $\mathbb{P}'$) is larger. As $\mathbb{N}'$ (or $\mathbb{P}'$) is used to fill the remaining cells in the SCMS (PSCMS) then M must be suitably large in order to have enough values for the given grid. Throughout this chapter the centre cell is always taken to be non-empty, hence the value M is known and the set $\mathbb{N}'$ (or $\mathbb{P}'$) is easily determined.

The concept of partial SCMS is introduced and different kinds of completability explored, with results given to aid an understanding of the relationship between completability on SCMS of order 3 and then general SCMS. This leads to the novel results of the cardinality of the minimal forced completable set and minimal critical set of a SCMS of order n, n odd. The concept of unavoidable sets is then defined for partial SCMS in general, with the concepts of proper and improper unavoidable sets, and their forms, defined. A full classification of unavoidable sets on minimum PSCMS of order 5 is then given with the minimum cardinality of each form found.

3.2 Completability of Partial SCMS

Definition 3.3. *A grid of order n is termed* ***partial*** *if between 1 and $n^2 - 1$ cells are non-empty. The set of tuples (i, j) of the non-empty cells is denoted H_n. When the non-empty cells have values such that the partial grid is* ***completable*** *to a SCMS (Definition 1.7) then the set of triples (i, j, a_{ij}) is denoted $\mathbf{H}_n$ and termed a* ***partial SCMS****.*

An example of a partial PSCMS of order 3 is given in Figure 3.1(a).

Definition 3.4. *Let $\mathbb{A}_n^{\mathbf{H}}$ be the set of all SCMS to which $\mathbf{H}_n$ can be completed. Hence, if $|\mathbb{A}_n^{\mathbf{H}}| = 1$ then $\mathbf{H}_n$ is* ***uniquely completable****, and if $|\mathbb{A}_n^{\mathbf{H}}| > 1$ then $\mathbf{H}_n$ is* ***multiply completable****.*

That is, if there is only one way of completing the grid then it is uniquely completable, and if there is more than one way of completing the grid then it is multiply completable. Only partial grids which are partial SCMS, $\mathbf{H}_n$, are considered, i.e. the partial grid is completable to a SCMS. Operations can be defined for assigning values, a_{ij}, to empty cells.

Definition 3.5. *A cell is* ***row-completable*** *(or* ***column-completable****) if $n - 1$ cells are non-empty in the given row (or column). A cell (i, j) is* ***pairwise-completable*** *if its paired cell $(\bar{i}, \bar{j})$ is non-empty.*

Definition 3.6. *An empty cell (i, j) in a partial SCMS, $\mathbf{H}_n$, that can be filled uniquely using a row/column/pairwise completable operation is referred to as* ***forced****.*

Definition 3.7. *A partial SCMS, $\mathbf{H}_n$, is termed* ***strongly completable*** *if every empty cell becomes forced at some iteration of completion, otherwise it is* ***weakly completable****.*

In order to understand critical sets, the completability of partial SCMS is explored.

Lemma 3.8. *A strongly completable partial SCMS,* $\mathbf{H}_n$*, is uniquely completable. A weakly completable partial SCMS,* $\mathbf{H}_n$*, is either uniquely or multiply completable.*

Proof. From Definition 3.7, all empty cells in a strongly completable partial SCMS, $\mathbf{H}_n$, are forced, hence there is only one way to complete the grid. A weakly completable partial SCMS, $\mathbf{H}_n$, has at least one cell which is not forced. For specific partial SCMS, one or more given cells that are not forced can be completed in more than one way, and one such example is given in Figure 3.2(c). For other specific partial SCMS having one or more given cells that are not forced, all such cells are completable in only one way, and one such example is given in Figure 3.2(a). □

A strongly completable grid is one which can be considered a valid puzzle as one or more logical deductions can be made at every iteration in order to always be able to uniquely fill one cell until the grid is complete. A weakly completable grid, even if it is uniquely completable, is not here considered a valid puzzle as there is trial and error involved in its completion.

Recall, from Chapter 1, that grids of order 3 are trivial SCMS. Attention will be given first to SCMS of order 3 before looking at SCMS of order 5 and of order n. Completability will be important for finding critical sets of SCMS of odd order, all of which will have a subsquare of order 3.

Lemma 3.9. *A completable partial SCMS of order 3,* $\mathbf{H}_3$*, with two non-empty, non-paired cells in the border,* B_3*, including at least one corner cell, is strongly completable and hence uniquely completable.*

Proof. Without loss of generality, assume the completable partial SCMS includes $(1, 1, a_{11})$ and one other non-paired triple, (i, j, a_{ij}), in addition to the centre cell triple $(2, 2, a_{22})$.

If $i = 1$ (or $j = 1$), then the third cell in the row (or column) is forced completable and the row (or column) is completed, and hence $(3, k, a_{3,k})$ for $k = 1, \ldots, 3$ are pairwise completable (or likewise for $(l, 3, a_{l,3})$ for $l = 1, \ldots, 3$). The remaining two cells are forced.

If $i \neq 1$ and $j \neq 1$ then the paired cell of a_{ij} is in either row 1 or column 1 and forced. Hence, the above argument applies. Hence, the grid is strongly completable and from Lemma 3.8 is uniquely completable. □

Corollary 3.10. *A completable partial SCMS of order 3,* $\mathbf{H}_3$*, with two non-empty non-paired cells in the border,* B_n*, neither of which is a corner cell, is weakly and uniquely completable.*

Proof. The paired cells of the two non-empty cells in the border are forced, leaving just the corner cells empty. When all corner cells are empty no cell is forced, therefore the grid is not strongly completable. However, there is only one way of completing the four corners to satisfy the magic constant constraint of a Magic Square. □

Lemma 3.11. *A completable partial SCMS of order 3,* $\mathbf{H}_3$*, with fewer than two non-empty cells in the border,* B_3*, is weakly completable and multiply completable.*

Proof. Consider the case of a single non-empty corner cell, in addition to the centre cell. Without loss of generality, assume the completable partial SCMS includes $(1, 1, a_{11})$, hence $(3, 3, a_{33})$ is pairwise completable and no other cell is immediately forced. There exists $a_{12}, a_{13} \in \mathbb{N}'$ such that $\sum_{j=1}^{3} a_{1j} = 3M$ and a_{12}, a_{13} are not paired, and $a_{21}, a_{31} \in \mathbb{N}'$ such that $\sum_{i=1}^{3} a_{i1} = 3M$, a_{21}, a_{31} are not paired and $a_{31} = \bar{a}_{13}$. As $a_{12} + a_{13} = a_{21} + a_{31}$, then another completion of the partial SCMS exists in which both the paired values in positions $(1, 3)$ and $(3, 1)$ are permuted, and the non-paired values in positions $(1, 2)$ and $(2, 1)$ are permuted. Hence there are at least two completions, and therefore the SCMS is multiply completable.

Consider the case of a single non-empty, non-corner cell, in addition to the centre cell. Without loss of generality assume the completable partial SCMS includes $(1, 2, a_{12})$, hence $(3, 2, a_{32})$ is pairwise completable and no other cell is immediately forced. There must exist two complement sets of three values in $\mathbb{N}'$ summing to $3M$ in order to form the first column and the third column, such that the column and row sums are satisfied. Each of these can be placed in either the first or the third column, and hence there are always at least two completions and therefore the SCMS is multiply completable.

With all cells empty, other than the centre cell, there are no fewer solutions than in the cases above. □

Lemma 3.11 is illustrated for $M = 251$ in Figure 3.1, with a multiply completable partial PSCMS of order 3 shown in Figure 3.1(a) and its two equivalent completions shown in Figures 3.1(b) and 3.1(c).

461		
	251	

(a) Two Non-empty Cells in a Grid of Order 3

461	23	269
59	251	443
233	479	41

(b) First Completion of Figure 3.1(a)

461	59	233
23	251	479
269	443	41

(c) Second Completion of Figure 3.1(a)

Figure 3.1: *A Mutliply Completable Partial PSCMS of Order 3 and its Two Equivalent Completions*

Corollary 3.12. *For a completable partial SCMS of order 3 to be strongly completable it must contain at least two non-empty, non-paired cells, including at least one corner cell in addition to the centre cell.*

Proof. Follows directly from Lemmas 3.9 and 3.11 and Corollary 3.10. □

Now again consider PSCMS of order 5. From Lemma 3.8 a weakly completable partial SCMS, $\mathbf{H}_n$, is either uniquely or multiply completable, and an example of each case is given in Example 3.13 for $n = 5$ with $M = 251$. A weakly and uniquely completable partial PSCMS is given in Figure 3.2(a) with its unique completion given in Figure 3.2(b). A weakly and multiply completable partial PSCMS is given in Figure 3.2(c) with its two completions given in Figures 3.2(b) and 3.2(d).

Example 3.13. *Figures 3.2(a) and 3.2(c) each show a partial PSCMS, $\mathbf{H}_5$, with $M = 251$ and a non-empty subsquare of order 3. In Figure 3.2(a) the triples* $(1, 1, 419)$, $(1, 2, 71)$,

$(1,3,311)$, $(2,1,11)$, $(3,1,113)$ and their complement pairs are non-empty and highlighted. The empty cells A, B, C and their complements have only one completion, which is given in Figure 3.2(b). In Figure 3.2(c) the triples $(1,1,419)$, $(1,2,71)$, $(2,1,11)$, $(3,1,113)$ and their complement pairs are non-empty and highlighted. The empty cells A, B, C, D and their complements have two possible completions given in Figures 3.2(b) and 3.2(d).

419	71	311	A	B
11	461	23	269	491
113	59	251	443	389
C	233	479	41	$\bar{C}$
$\bar{B}$	431	191	$\bar{A}$	83

(a) A Uniquely Completable Partial PSCMS, $\mathbf{H}_5$

419	71	311	401	53
11	461	23	269	491
113	59	251	443	389
263	233	479	41	239
449	431	191	101	83

(b) Unique Completion of the Partial PSCMS Given in Figure 3.2(a) and One Completion of the Partial PSCMS Given in Figure 3.2(c)

419	71	D	A	B
11	461	23	269	491
113	59	251	443	389
C	233	479	41	$\bar{C}$
$\bar{B}$	431	$\bar{D}$	$\bar{A}$	83

(c) A Multiply Completable Partial PSCMS, $\mathbf{H}_5$

419	71	401	311	53
11	461	23	269	491
113	59	251	443	389
263	233	479	41	239
449	431	101	191	83

(d) Second Completion of the Partial PSCMS Given in Figure 3.2(c), Equivalent to Figure 3.2(b)

Figure 3.2: *Two Partial PSCMS, $\mathbf{H}_5$, and Two Completions*

Theorem 3.14. *For any partial SCMS, $\mathbf{H}_n$, of order n, n odd:*

(1) If $\mathbf{H}_n$ is strongly completable then it is uniquely completable.

(2) If $\mathbf{H}_n$ is multiply completable then it is weakly completable.

(3) If $\mathbf{H}_n$ is weakly completable then it is either uniquely completable or multiply completable.

(4) If $\mathbf{H}_n$ is uniquely completable then it is either strongly completable or weakly completable.

Proof. (1) This follows immediately from Lemma 3.8.

(2) If a grid is multiply completable then it cannot be strongly completable, hence it is weakly completable.

(3) This follows immediately from Lemma 3.8.

(4) From conditions (1) and (3), as both strongly and weakly completable can imply uniquely completable.

□

3.3 Forced Completable Sets and Critical Sets of Partial SCMS

The above exploration of completability is now used to determine bounds on the cardinality of critical sets of partial SCMS, and the exact cardinality for minimum PSCMS of order 5.

Definition 3.15. *A* ***forced completable set*** *is a set which contains the non-empty cells of a partial SCMS, $\mathbf{H}_n$, that is strongly completable. Let F_n denote the set of cells of a forced completable set and $\mathbf{F}_n$ denote the set of triples defining that forced completable set.*

Hence, for clarity, an alternative definition is:

Definition 3.16. *A forced completable set $\mathbf{F}_n$ defines a partial SCMS that is uniquely completable to $\mathbf{A}_n$ (i.e. $|\mathbb{A}_n^{\mathbf{F}}| = 1$) using only row/column/pairwise completion operations.*

Definition 3.17. *A* ***minimal forced completable set*** *of a SCMS of order n, denoted as $\mathbf{F}_n^{min}$, is a forced completable set of minimum cardinality.*

A given grid, $\mathbf{A_n}$, may have more than one minimal forced completable set. Two examples of partial grids specified by minimal forced completable sets, $\mathbf{F}_5^{min}$, are given in Figures 3.3(a) and 3.3(b) which are uniquely completable to the grid in Figure 3.3(c). It is shown that these partial grids are specified by a minimal set in Theorem 3.26.

311	113		401	
	461	23		239
		251		
				53
		491		

(a) First Partial PSCMS Specified by a Minimal Forced Completable Set

		11		419
			269	
		251		353
449			41	
	389			191

(b) Second Partial PSCMS Specified by another Minimal Forced Completable Set

311	113	11	401	419
263	461	23	269	239
149	59	251	443	353
449	233	479	41	53
83	389	491	101	191

(c) Unique Completion of the Partial PSCMS in (a) and (b)

Figure 3.3: *Two Partial PSCMS,* $\mathbf{H}_5$*, Specified by Minimal Forced Completable Sets,* $\mathbf{F}_5^{min}$*, and their Unique Completion,* $\mathbf{A}_5$

Similar to the definition of a critical set of a Latin Square, given in Section 1.3 [9], a critical set of a SCMS is here defined.

Definition 3.18. *A* ***critical set****,* $\mathbf{V}_n$*, of a SCMS,* $\mathbf{A}_n$*, of order* n*, is a set* $\mathbf{V}_n = \{(i, j, a_{ij}) \mid i, j \in \{1, \ldots, n\},\ a_{ij} \in \mathbb{N}'\}$ *such that:*

(1) $\mathbf{A}_n$ *is the only SCMS of order* n *which has entry* a_{ij} *in position* (i, j) *for all* $(i, j, a_{ij}) \in \mathbf{V}_n$*.*

(2) no proper subset of $\mathbf{V}_n$ *satisfies (1).*

Definition 3.19. *Let V_n denote the set of cells of a critical set, $\mathbf{V}_n$, of a SCMS, $\mathbf{A}_n$, where $\mathbf{V}_n$ defines a partial SCMS, and let $\mathbb{V}_n^{\mathbf{A}}$ be the set of all critical sets of $\mathbf{A}_n$.*

Definition 3.20. *A critical set $\mathbf{V}_n$ is a partial SCMS that is uniquely completable to a SCMS $\mathbf{A}_n$, i.e. $|\mathbb{A}_n^{\mathbf{V}}| = 1$, and is such that if any triple is removed from $\mathbf{V}_n$ the resulting partial SCMS is multiply completable, i.e. for any $\mathbf{H}_n \subset \mathbf{V}_n$, $|\mathbb{A}_n^{\mathbf{H}}| > 1$.*

Definition 3.21. *A critical set, $\mathbf{V}_n$, is termed a **strong critical set** if it is also a forced completable set, else it is a **weak critical set**.*

Definition 3.22. *A **minimal critical set** of a SCMS of order n, denoted as $\mathbf{V}_n^{min}$, is a critical set of minimum cardinality.*

Consider, $\exists \mathbf{V}_n^{min} \in \mathbb{V}_n^{\mathbf{A}}$ such that $\forall \mathbf{V}_n \in \mathbb{V}_n^{\mathbf{A}}$, $|\mathbf{V}_n^{min}| \leq |\mathbf{V}_n|$, and hence $\mathbf{V}_n^{min}$ is a minimal critical set of $\mathbf{A}_n$. There may be more than one minimal critical set of a given SCMS.

Lemma 3.23. *A critical set, $\mathbf{V}_n$, describes a partial SCMS, $\mathbf{H}_n$, which is uniquely completable.*

Proof. This follows immediately from Definition 3.18, using strong completability or weak completability. □

A comparison of the cardinality of the minimal forced completable set, $\mathbf{F}_n^{min}$, and the cardinality of the minimal critical set, $\mathbf{V}_n^{min}$, of a SCMS of order n, n odd, is now given and the cardinality of the minimal forced completable set for a SCMS of order n, n odd, is calculated. Bounds are given below for the cardinality of a minimal critical set of a SCMS of order n, n odd.

Lemma 3.24. $|\mathbf{V}_n^{min}| \leq |\mathbf{F}_n^{min}|$

Proof. For any given SCMS, $\mathbf{A}_n$, any minimal forced completable set, $\mathbf{F}_n^{min}$, is a strong critical set, $\mathbf{V}_n$, if by removing an element from $\mathbf{F}_n^{min}$ the partial SCMS, $\mathbf{H}_n$, is no longer uniquely completable to $\mathbf{A}_n$. If no set of smaller size than all $|\mathbf{F}_n^{min}|$ forms a critical set on the SCMS then $|\mathbf{V}_n^{min}| = |\mathbf{F}_n^{min}|$. If for any $\mathbf{F}_n^{min}$, an element can be removed to form a weak critical set for $\mathbf{A}_n$, then $|\mathbf{V}_n^{min}| < |\mathbf{F}_n^{min}|$. □

Lemma 3.25. *If a SCMS of order n, n odd, exists then the cardinality of its minimal forced completable set is given by the recurrence* $|\mathbf{F}_n^{min}| = 2n - 4 + |\mathbf{F}_{n-2}^{min}|$ *where* $|\mathbf{F}_1^{min}| = 1$.

Proof. The centre cell M is always non-empty hence $|\mathbf{F}_1^{min}| = 1$, and both $\mathbb{N}'$ and S_n are known. Firstly, consider a SCMS of order 3, from Corollary 3.12; two non-paired cells, including at least one corner cell must be non-empty in the border for the grid to be strongly completable, hence $|\mathbf{F}_3^{min}| = 3$, which satisfies the given recurrence. Assume the recurrence is true for some $n = k$, $k > 3$ and odd, $|\mathbf{F}_k^{min}| = 2k - 4 + |\mathbf{F}_{k-2}^{min}|$. Now consider the case $n = k + 2$, where k cells are added in each of the new border rows and $k - 2$ further cells are added in each new border column. Of these, $(2(k+2) - 4)$ must be non-empty else there would be empty cells that were not forced completable, and hence $(2(k + 2) - 4)$ triples are added to $\mathbf{F}_k^{min}$, hence $|\mathbf{F}_{k+2}^{min}| = 2(k + 2) - 4 + |\mathbf{F}_k^{min}|$. By induction the recurrence holds for any n odd. □

From the recurrence the integer sequence obtained is A058331 [11], and is discussed by the current author in [34].

Theorem 3.26. *If a SCMS of order n, n odd, exists then the size of its minimal forced completable set is* $|\mathbf{F}_n^{min}| = \frac{1}{2}(n^2 - 2n + 3)$.

Proof. For j odd, $j = (2i + 1)$ where $i = 1, \dots, \frac{n-1}{2}$, the size of the set of non-empty cells, $\mathbf{Z}_j$, for the outer border where $j = n$ and for each border of an order j subsquare $j \geq 3$ is $|\mathbf{Z}_j| = 2j - 4$. Hence, $|\mathbf{Z}_j| = 2(2i+1) - 4 = 4i - 2$. Hence, $|\mathbf{Z}_n| = 4\sum_{i=1}^{\frac{n-1}{2}} i - \sum_{i=1}^{\frac{n-1}{2}} 2$. Since $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, then $4\sum_{i=1}^{\frac{n-1}{2}} i = \frac{4}{2}(\frac{n-1}{2})(\frac{n-1}{2}+1)$ and $\sum_{i=1}^{\frac{n-1}{2}} 2 = n-1$ giving $|\mathbf{Z}_n| = \frac{1}{2}(n^2 - 2n + 1)$. Since the centre cell is not included in a border but is always non-empty, then $|\mathbf{F}_n^{min}| = \sum_{j=3}^{n} \mathbf{Z}_j + 1$ hence $|\mathbf{F}_n^{min}| = \frac{1}{2}(n^2 - 2n + 1) + 1 = \frac{1}{2}(n^2 - 2n + 3)$. □

Theorem 3.27. *A partial grid of order n, n odd, is a minimal forced completable set if the centre cell is non-empty and $2j-4$ cells are non-empty in the outer border where $j=n$ and in the border of each subsquare of order j, $j=3,5,\dots,n-2$, with at most one cell of any pair non-empty and at least one non-empty corner cell in each border.*

Proof. From the proof of Lemma 3.25, the minimal forced completable set of a SCMS of order n must contain the centre cell and within the border of each subsquare of order j, $j=3,5,\dots,n-2$, and within the outer border ($j=n$) it must contain $2j-4$ non-empty cells, out of $4j-4$ border cells.

Suppose all corner cells are empty, the requirement for $2n-4$ cells being non-empty necessarily means that all non-corner cells in one row and one column are non-empty. All cells are pairwise completable except the corner cells and given there are two empty cells in each row and column, the four corner cells are not forced. Hence, it is required that there is at least one non-empty corner cell in each border for the grid to be strongly completable. □

Theorem 3.28. *Given a SCMS of order n, n odd, $n \geq 5$, the size of the minimal critical set, $|\mathbf{V}_n^{min}|$, satisfies $\frac{1}{2}(n^2-4n+9) \leq |\mathbf{V}_n^{min}| \leq \frac{1}{2}(n^2-2n+3)$.*

Proof. First consider the upper bound of the size of the minimal critical set, from Lemma 3.24 $|\mathbf{V}_n^{min}| \leq |\mathbf{F}_n^{min}|$ and from Theorem 3.26 $|\mathbf{F}_n^{min}| = \frac{1}{2}(n^2-2n+3)$.

Now consider the SCMS of order 3, from Lemma 3.9 and Corollary 3.10 there are two non-empty cells in the border for the grid to be uniquely completable. Hence a critical set contains no fewer than two triples, and with the inclusion of the centre cell value contains three triples. When $n=3$, $|\mathbf{F}_3^{min}|=3$ hence there are exactly three non-empty cells for the minimal critical set, $|\mathbf{V}_3^{min}|=3$.

For j odd, $j=(2i+3)$ where $i=1,\dots,\frac{n-3}{2}$, the borders of order j of the subsquares, and the outer border ($j=n$), of the SCMS of order n each have $4j-4$ cells. For an SCMS of order n, $n>3$, $2j-6$ non-empty cells are required in each border for the SCMS to be uniquely completable, else it is multiply completable due to the permutation operations in Table 1.1.

Hence, $|\mathbf{Z}_j| = 2j - 6 = 2(2i+3) - 6 = 4i$. Hence, $|\mathbf{Z}_n| = 4\sum_{i=1}^{\frac{n-3}{2}} i$. Since $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, then $4\sum_{i=1}^{\frac{n-3}{2}} i = \frac{4}{2}(\frac{n-3}{2})(\frac{n-3}{2}+1)$, giving $|\mathbf{Z}_n| = \frac{1}{2}(n^2 - 4n + 3)$.

The subsquare of order 3 has three non-empty cells, and hence $|\mathbf{V}_n^{min}| = \sum_{j=5}^{n} \mathbf{Z}_j + 3$ and hence $|\mathbf{V}_n^{min}| = \frac{1}{2}(n^2 - 4n + 9)$. □

Construction 3.29. *Take an empty grid of order 3 and fill in the centre cell to form a valid partial SCMS. Secondly place one value in a corner to form a valid partial SCMS. Thirdly, place a further value in any cell that is not paired with that of the previously placed value to form a valid partial SCMS.*

Theorem 3.30. *A forced completable set of a SCMS of order 3, that is also a minimal critical set, is specified by a partial grid obtained by Construction 3.29.*

Proof. Using Construction 3.29 the number of non-empty cells is 3 which satisfies $\frac{1}{2}(n^2 - 2n + 3)$ which is both the size of the minimal forced completable set and the upper bound of the size of the minimal critical set. If either of the values in the border are removed then the resulting grid is multiply completable, and hence the remaining triples do not form a critical set nor a forced completable set. □

Without loss of generality, let the two non-empty cells of a partial SCMS of order 3, specified in Construction 3.29, be the cells $(1,1)$ and $(1,2)$ illustrated using a PSCMS in Figure 3.4(a) with its completion given in Figure 3.4(b). Hence, $|\mathbf{V}_3^{min}| = 3$ which satisfies the upper bound of Theorem 3.28.

461	23	
	251	

(a) Three Non-Empty Cells in a Grid of Order 3

461	23	269
59	251	443
233	479	41

(b) Unique Completion to Figure 3.4(a)

Figure 3.4: *A Partial PSCMS,* $\mathbf{H}_3$*, Specified by a Minimal Critical Set,* $\mathbf{V}_3^{min}$ *and its Unique Completion,* $\mathbf{A}_3$

The cardinality of the minimal critical set, $|\mathbf{V}_n^{min}|$, of a PSCMS of order n, n odd, with known M and hence known $\mathbb{P}'$, can be determined. For a specific example, the minimum PSCMS of order 5 have been taken from Chapter 2, [35] and the cardinality of their minimal critical set is given.

Theorem 3.31. *The cardinality of the minimal critical set of a minimum PSCMS of order 5 is* $|\mathbf{V}_5^{min}| = 7$*, which satisfies* $\frac{1}{2}(n^2 - 4n + 9)$*, the lower bound of Theorem 3.28.*

Proof. Suppose for contradiction there are six non-empty cells, including the centre cell. It can be clearly seen using the permutation operations in Table 1.1 that, irrespective of which cells are non-empty, the remaining values can be placed in the empty cells to complete the grid in multiple ways. An example of a critical set of size 7 of a minimum PSCMS of order 5 is given in Figure 3.5, and hence the cardinality of the minimal critical set of a minimum PSCMS of order 5 is 7. □

	71	311		
11	461	23		
113		251		

(a) A Grid Specified by a Minimal Critical Set of a Minimum PSCMS of Order 5

419	71	311	401	53
11	461	23	269	491
113	59	251	443	389
263	233	479	41	239
449	431	191	101	83

(b) Unique Completion of the Partial PSCMS in (a)

Figure 3.5: *A Partial PSCMS, $\mathbf{H}_5$, Specified by a Minimal Critical Set, $\mathbf{V}_5^{min}$, of a Minimum PSCMS and its Unique Completion, $\mathbf{A}_5$*

In the literature on Latin Squares, the concepts of strong and weak critical sets are presented as well as a definition of a 'totally weak' critical set where no cell is forced initially [17]. The concept of 'totally weak' does not apply to SCMS, as to avoid any empty cell being forced the only cells that could be non-empty are the centre cell, or both values of given paired cells. Given a centre cell, the completion of any single cell results in the availability of a forced cell completion by a pairwise-completion operation. A set containing only the centre cell is not a critical set as the partial grid so defined is not uniquely completable. If both values in paired cells are non-empty and the grid is uniquely completable, then one of the values can be removed and the grid is still uniquely completable, and so the filled cells do not form a critical set. Hence, the concept of a totally weak critical set does not exist for any SCMS.

3.4 Unavoidable Sets of SCMS

Recall that H_n denotes the non-empty cells of the partial SCMS, $\mathbf{H}_n$. Correspondingly denote by H'_n the set of empty cells.

As with Latin Squares, an **unavoidable set** of a SCMS is defined here (Definition 3.32) to be a set of tuples such that if all those cells are empty in a partial SCMS, $\mathbf{H}_n$, then $\mathbf{H}_n$ is multiply completable.

Let D_n denote the set of cells of an unavoidable set of a SCMS $\mathbf{A}_n$. (There is no corresponding set of triples, $\mathbf{D_n}$, as the values in the cells of D_n are not defined.) Hence, $D_n \subset A_n$, where A_n is the set of cells corresponding to $\mathbf{A}_n$.

Definition 3.32. *An unavoidable set, D_n, of a SCMS, $\mathbf{A}_n$, is such that the partial SCMS, $\mathbf{H}_n$, for which $H_n = A_n \setminus D_n$, is multiply completable, i.e. $|\mathbb{A}_{\mathbf{n}}^{\mathbf{H}}| > 1$ and either*

(1) the addition to $\mathbf{H}_n$ of any triple $(i, j, a_{i,j}) \in \mathbf{A}_n$ where $(i, j) \in D_n$, produces a partial SCMS, $\mathbf{G}_n$, which is uniquely completable; or

(2) the addition to $\mathbf{H}_n$ of any triple $(i, j, a_{i,j}) \in \mathbf{A}_n$ where $(i, j) \in D_n$, produces a partial SCMS, $\mathbf{G}_n$, which is either uniquely completable or multiply completable to grids that are equivalent.

A given SCMS can have more than one unavoidable set, and these sets can overlap. Strictly, unavoidable sets are sub-structures present in all of the SCMS to which a given partial grid completes. However, the remainder of this chapter will refer to unavoidable sets being present in partial SCMS.

Definition 3.33. *Some partial SCMS, $\mathbf{H}_n$, completable to any $\mathbf{A}_n \in \mathbb{A}_n^{\mathbf{H}}$, may contain cells that are forced completable and a number of unavoidable sets. Let the set of unavoidable sets of a partial SCMS, $\mathbf{H}_n$, be $\mathbb{D}_n^H$, and so $\bigcup_{D_n \in \mathbb{D}_n^H} D_n \subseteq H'_n$.*

This thesis does not address overlapping unavoidable sets as it is not of benefit in classifying the forms of unavoidable sets. Note that, analysis of partial grids possessing overlapping unavoidable sets is not possible without first having a classification of forms of unavoidable sets. Hence only partial SCMS with a set of empty cells that correspond to a single unavoidable set of all grids in $\mathbb{A}_{\mathbf{n}}^{\mathbf{H}}$ are considered.

Recall from critical sets and forced completable sets that whenever one half of a complement pair is included in a partial SCMS, its partner is immediately forced. For the purpose of the following analysis of unavoidable sets it is assumed that for any partial SCMS considered, either both triples of a given complement pair are included or neither. Hence any unavoidable set will be a collection of paired cells.

3.4.1 Proper and Improper Unavoidable Sets

In order for a partial SCMS to be completed as a puzzle it must be strongly completable. In this thesis a grid is considered a puzzle only if it is solvable using logical deductions and therefore a weakly, uniquely completable grid which requires some trial and error is not considered a puzzle. From Definition 3.32, condition (1) relates to a legitimate puzzle and is referred to as a **proper unavoidable set**. In this thesis, interest extends beyond the properties of SCMS relating to puzzles, as defined here, in particular as the analysis of PSCMS of order 5 explored non-equivalent completions (Section 2.2.3). Therefore the unavoidable sets relating to Definition 3.32 condition (2) are also of interest, and these patterns are referred to as **improper unavoidable sets**.

A complete classification of the unavoidable sets of minimum PSCMS of order 5 is given. There are only five forms, and these are detailed in Sections 3.4.2 and 3.4.3. Note that all figures in examples in these sections are formed from the minimum PSCMS of order 5 in Chapter 2, using logical reasoning and a knowledge of paired values.

Definition 3.34. *An unavoidable set is here considered a* ***proper unavoidable set*** *if it conforms to condition (1) of Definition 3.32.*

In such cases $\mathbb{A}_{\mathbf{n}}^{\mathbf{H}}$ contains multiple completions of the partial grid, $\mathbf{H}_n$, which are non-equivalent, equivalent or both, and the addition of a further triple constrains the completion to a unique grid in $\mathbb{A}_{\mathbf{n}}^{\mathbf{H}}$.

Definition 3.35. *An unavoidable set is here considered an* ***improper unavoidable set*** *if it conforms to condition (2) of Definition 3.32.*

In such cases $\mathbb{A}_{\mathbf{n}}^{\mathbf{H}}$ contains non-equivalent and equivalent completions of the partial grid $\mathbf{H}_n$, and the addition of a further triple constrains the completion either to a unique grid or to only multiple equivalent grids in $\mathbb{A}_{\mathbf{n}}^{\mathbf{H}}$.

Corollary 3.36. *Any two empty, non-corner cells in any single row or column of a border of a SCMS of order n, $n \geq 5$, (with their paired cells also empty) form a proper unavoidable set.*

Proof. From Table 1.1 the values in the paired cells in rows (columns) $2, \ldots, n-1$ of the grid can be permuted to form another valid completion, and hence the cells form an unavoidable set. □

A Normal SCMS of order 7 from [2] is used in Figure 3.6 to illustrate two separate proper unavoidable sets, the first in the border of order 7 (highlighted blue) and the second in the border of order 5 (highlighted grey).

46	1	2	3	42	41	40
45	35	13	14	32	31	5
44	34	28	21	26	16	6
7	17	23	25	27	33	43
11	20	24	29	22	30	39
12	19	37	36	18	15	38
10	49	48	47	8	9	4

Figure 3.6: *Normal SCMS of Order 7 Containing Two Unavoidable Sets*

Different patterns of empty cells are now explored on partial PSCMS of order 5, completable to minimum PSCMS, where the interest is in the minimum number of empty cells in a given form of unavoidable set (defined below).

There are two forms of proper unavoidable set (Definitions 3.37 and 3.42) and an additional three forms of improper unavoidable set (Definitions 3.46, 3.51, 3.56) that can be identified for SCMS of order 5. These are applied to the minimum PSCMS of order 5 given in Chapter 2. Forms 1 and 2, proper unavoidable sets (Definitions 3.37 and 3.42) define patterns of empty cells that are mutually exclusive. Likewise Forms 3, 4 and 5, improper unavoidable sets (Definitions 3.46, 3.51, 3.56), define patterns of empty cells that are mutually exclusive to each other and to those patterns of proper unavoidable sets. It is possible, that for multiple patterns of empty cells, the corresponding partial grids complete to the same PSCMS. That is, two partial PSCMS with different patterns of empty cells could have one or more of their completions in common.

Given a PSCMS with fixed prime M, $\mathbb{P}'$ denotes the set of all primes that can be paired to sum to $2M$. Recall from Chapter 2 the minimum PSCMS of order 5 has $M = 251$, $|\mathbb{P}'| = 26$ and the grids are split into types depending on the subsquare of order 3 and the primes in the border.

3.4.2 Proper Unavoidable Sets on Minimum PSCMS of Order 5

3.4.2.1 Form 1 Unavoidable Sets on Minimum PSCMS of Order 5

Definition 3.37. *Consider a partial PSCMS of order 5, $\mathbf{H}_5$, such that all empty cell tuples form a single unavoidable set and are located within the outer border. If all the multiple completions of $\mathbf{H}_5$, in $\mathbb{A}_5^{\mathbf{H}}$, use a single subset of $\mathbb{P}'$ and are equivalent using the permutations of paired cells in columns $2, 3, 4$ of the grid or the permutations of paired cells in rows $2, 3, 4$ of the grid (from Table 1.1) then the unavoidable set is defined to be of Form 1 denoted $D_5^{\mathbf{H},1}$.*

An unavoidable set of Form 1 is given in Figure 3.7 in a minimum PSCMS of order 5. A Form 1 unavoidable set is always a proper unavoidable set, and this is shown in Lemma 3.39. Example 3.38 illustrates a partial PSCMS, $\mathbf{H}_5$, multiply completable to minimum PSCMS of order 5, and its two equivalent completions, $|\mathbb{A}_5^{\mathbf{H}}| = 2$.

Example 3.38. *Figure 3.7 shows a partial PSCMS of order 5 with a Form 1 unavoidable set of size 4 in cells* $(1,2),(1,3),(5,2),(5,3)$ *and the two equivalent completions.*

419			401	53
11	461	23	269	491
113	59	251	443	389
263	233	479	41	239
449			101	83

(a) Form 1 Unavoidable Set in a Minimum PSCMS of order 5

419	71	311	401	53
11	461	23	269	491
113	59	251	443	389
263	233	479	41	239
449	431	191	101	83

(b) First Completion of (a)

419	311	71	401	53
11	461	23	269	491
113	59	251	443	389
263	233	479	41	239
449	191	431	101	83

(c) Second Completion of (a)

Figure 3.7: *Form 1 Unavoidable Set of Size 4 in a Minimum PSCMS of Order 5 with its Two Completions*

The completions shown in Figures 3.7(b) and 3.7(c) are equivalent, using the same 24 border pairs, and are minimum PSCMS of order 5 of type 1A using complement sets of primes A1 and A3 from Table 2.2 in Chapter 2. The addition of any triple to the partial PSCMS in Figure 3.7(a) determines a unique completion, and hence the empty cells describe a proper unavoidable set.

Lemma 3.39. *An unavoidable set of Form 1 is a proper unavoidable set.*

Proof. A Form 1 unavoidable set satisfies Definition 3.34 and hence condition (1) of Definition 3.32. □

Theorem 3.40. *If a partial PSCMS, $\mathbf{H}_n$, contains an unavoidable set of Form 1, $D_n^{\mathbf{H},1}$, then $|D_n^{\mathbf{H},1}| = 4$.*

Proof. Consider first a partial PSCMS with fewer than four empty cells, it is uniquely completable. Next consider a partial PSCMS with exactly four empty cells, the addition of any triple determines a unique completion. Finally, consider a partial PSCMS with more than four empty cells, the addition of a triple does not guarantee a uniquely completable grid. □

Corollary 3.41. *If a partial PSCMS, $\mathbf{H}_5$, completable to a minimum PSCMS, contains an unavoidable set of Form 1, $D_5^{\mathbf{H},1}$, then $|D_5^{\mathbf{H},1}| = 4$.*

Proof. Follows immediately from Theorem 3.40. □

3.4.2.2 Form 2 Unavoidable Sets on Minimum PSCMS of Order 5

Definition 3.42. *Consider a partial PSCMS of order 5, $\mathbf{H}_5$, such that all empty cell tuples form a single unavoidable set and are located within the outer border. If all the multiple completions of $\mathbf{H}_5$, in $\mathbb{A}_5^{\mathbf{H}}$, are non-equivalent and each completion uses a different subset of $\mathbb{P}'$, then the empty cell tuples form an unavoidable set of Form 2 denoted $D_5^{\mathbf{H},2}$.*

An unavoidable set of Form 2 is given in Example 3.43 in a minimum PSCMS of order 5, this is a minimal unavoidable set of Form 2 and is a proper unavoidable set, and this is shown in Lemma 3.44. If a partial PSCMS of order 5 has an unavoidable set of Form 2, then there are multiple completions and these are all non-equivalent. Example 3.43 illustrates a partial PSCMS, $\mathbf{H}_5$, multiply completable to minimum PSCMS of order 5, and its two non-equivalent completions, $|\mathbb{A}_5^{\mathbf{H}}| = 2$.

Example 3.43. *Figure 3.8 shows a partial PSCMS of order 5 with a Form 2 unavoidable set of size 6 in cells $(1,4),(1,5),(4,1),(4,5),(5,1),(5,4)$ and the two non-equivalent completions.*

311	11	113		
149	461	23	269	353
263	59	251	443	239
	233	479	41	
	491	389		191

(a) Minimal Form 2 Unavoidable Set in a Minimum PSCMS of order 5

311	11	113	419	401
149	461	23	269	353
263	59	251	443	239
431	233	479	41	71
101	491	389	83	191

(b) First Completion of (a)

311	11	113	401	419
149	461	23	269	353
263	59	251	443	239
449	233	479	41	53
83	491	389	101	191

(c) Second Completion of (a)

Figure 3.8: *Minimal Form 2 Unavoidable Set of Size 6 in a Minimum PSCMS of Order 5 with its Two Completions*

The completions shown in Figures 3.8(b) and 3.8(c) are non-equivalent and use different subsets of $\mathbb{P}'$. Both of these completions are minimum PSCMS of order 5, the first completion is of type 1C using complement sets of primes C3 and C4 from Table 2.4 and the second completion is of type 1B using complement sets of primes B1 and B3 from Table 2.3. The addition of any triple to the partial PSCMS in Figure 3.8(a) determines a unique completion, and hence the empty cells describe a proper unavoidable set.

Lemma 3.44. *An unavoidable set of Form 2 is a proper unavoidable set.*

Proof. A Form 2 unavoidable set satisfies Definition 3.34 and hence condition (1) of Definition 3.32. □

Theorem 3.45. *If a partial PSCMS,* $\mathbf{H}_5$*, completable to a minimum PSCMS, contains an unavoidable set of Form 2,* $D_5^{\mathbf{H},2}$*, then* $|D_5^{\mathbf{H},2}| \geq 6$*, and partial PSCMS of order 5 exist that* *contain such unavoidable sets of size 6.*

Proof. Consider first a partial PSCMS, completable to a minimum PSCMS of order 5, with fewer than six empty cells. Assume for contradiction that the grid has four empty cells then either it is uniquely completable, and hence the cells do not form an unavoidable set, or it is a Form 1 unavoidable set. Now assume that the grid has any fewer than four empty cells, then it does not contain an unavoidable set as the grid is uniquely completable. An example of an unavoidable set, $D_5^{\mathbf{H},2}$, for which $|D_5^{\mathbf{H},2}| = 6$ is given in Figure 3.8. □

3.4.3 Improper Unavoidable Sets on Minimum PSCMS of Order 5

3.4.3.1 Form 3 Unavoidable Sets on Minimum PSCMS of Order 5

Definition 3.46. *Consider a partial PSCMS of order 5, $\mathbf{H}_5$, such that all empty cell tuples form a single unavoidable set and are located in both the outer border and the subsquare of order 3. For the empty cell tuples to form an unavoidable set of Form 3, denoted $D_5^{\mathbf{H},3}$, then:*

(1) all completions of $\mathbf{H}_5$, in $\mathbb{A}_5^{\mathbf{H}}$, that use the same subset of $\mathbb{P}'$ are equivalent; and

(2) all completions of $\mathbf{H}_5$, in $\mathbb{A}_5^{\mathbf{H}}$, that use different subsets of $\mathbb{P}'$ are non-equivalent.

It follows that a partial PSCMS, completable to a minimum PSCMS of order 5, having a single unavoidable set of Form 3 will have multiple equivalent and non-equivalent completions and trivially all completions that have different subsquares of order 3 are non-equivalent. Figure 3.9(a) shows a minimal unavoidable set of Form 3 in a minimum PSCMS of order 5. If a minimum PSCMS of order 5 has a Form 3 unavoidable set then there are empty cells in both the border of order 3 and the border of order 5. The fixing of any empty cell either gives a uniquely completable partial PSCMS or it gives a partial PSCMS where all completions are equivalent. Example 3.47 illustrates a partial PSCMS, $\mathbf{H}_5$, multiply completable to minimum PSCMS of order 5, and its two non-equivalent completions. The multiple equivalent completions are not given.

Example 3.47. *Figure 3.9 shows a partial PSCMS of order 5 with a Form 3 unavoidable set of size 12 in cells* $(1,3),(1,5),(2,2),(2,3),(2,4),(3,1),(3,5),(4,2),(4,3),(4,4),(5,1),(5,3)$ *and the two non-equivalent completions.*

353	11		401	
191				311
	59	251	443	
389				113
	491		101	149

(a) Minimal Form 3 Unavoidable Set in a Minimum PSCMS of order 5

353	11	41	401	449
191	431	83	239	311
269	59	251	443	233
389	263	419	71	113
53	491	461	101	149

(b) First Completion of (a); note each added complement pair is given a distinct colour

353	11	71	401	419
191	461	23	269	311
239	59	251	443	263
389	233	479	41	113
83	491	431	101	149

(c) Second Completion of (a); note each added complement pair is given a distinct colour, using the same colours as in (b) where appropriate.

Figure 3.9: *Minimal Form 3 Unavoidable Set of Size 12 in a Minimum PSCMS of Order 5 with its Two Non-Equivalent Completions*

The completions shown in Figures 3.9(b) and 3.9(c) are non-equivalent and use different subsets of $\mathbb{P}'$. Both of these completions are minimum PSCMS of order 5, the first completion is of type 2H using complement sets of primes H1 and H2 from Table 2.5 and the second completion is of type 1C using complement sets of primes C1 and C2 from Table 2.5. The addition of any triple to the partial PSCMS in Figure 3.9(a) determines either a unique completion or multiple equivalent completions, and hence the empty cells describe an improper unavoidable set.

Lemma 3.48. *An unavoidable set of Form 3 is an improper unavoidable set.*

Proof. A Form 3 unavoidable set satisfies Definition 3.35 and hence condition (2) of Definition 3.32. □

Example 3.49 repeats the partial PSCMS from Figure 3.9(a) and illustrates the outcome of the addition of two different triples, the first producing a partial PSCMS that is completable to two equivalent grids and the second producing a uniquely completable partial PSCMS. Hence the unavoidable set is improper.

Example 3.49. *Figure 3.10(a) shows a partial PSCMS, completable to a minimum PSCMS of order 5, with a Form 3 unavoidable set of size 12. Figure 3.10(b) shows, in yellow, the triple (1,3,41) added to (a) and the two equivalent completions are given in Figures 3.10(c) and 3.10(d). Figure 3.10(e) shows, in yellow, the triple (2,2,431) added to (a) and the one completion is given in Figure 3.10(f).*

353	11		401	
191				311
	59	251	443	
389				113
	491		101	149

(a) Multiply Completable Partial PSCMS with a Form 3 Improper Unavoidable Set

353	11	41	401	
191				311
	59	251	443	
389				113
	491		101	149

(b) Placing the Triple (1,3,41) into (a)

353	11	41	401	449
191	431	83	239	311
269	59	251	443	233
389	263	419	71	113
53	491	461	101	149

(c) First Completion of (b)

353	11	41	401	449
191	263	419	71	311
269	59	251	443	233
389	431	83	239	113
53	491	461	101	149

(d) Second Completion of (b) Equivalent to (c)

353	11		401	
191	431			311
	59	251	443	
389				113
	491		101	149

(e) Placing the triple (2,2,431) into (a)

353	11	41	401	449
191	431	83	239	311
269	59	251	443	233
389	263	419	71	113
53	491	461	101	149

(f) Unique Completion of (e) (which is also Equal to First Completion of (b))

Figure 3.10: *Demonstration of Placing Two Different Triples in a Form 3 Unavoidable set*

Theorem 3.50. *If a partial PSCMS, $\mathbf{H}_5$, completable to a minimum PSCMS of order 5, contains a Form 3 improper unavoidable set, $D_5^{\mathbf{H},3}$, then $|D_5^{\mathbf{H},3}| \geq 12$, and partial PSCMS of order 5 exist that contain such unavoidable sets of size 12.*

Proof. There are exactly two possible subsquares for the minimum PSCMS of order 5 and these differ by three pairs. Hence, in order for the subsquare to have more than one unique completion there must be six cells empty in the subsquare of order 3. If there are fewer than four empty cells (two pairs) in the border then it is forced completable and hence the border is uniquely completable and therefore the grid is uniquely completable.

Recall from Section 2.2.2 for the minimum PSCMS of order 5 $|\mathbb{P}'| = 26$, so each grid uses twelve pairs out of a possible thirteen and hence each subset can only differ by one pair. If there are exactly four empty cells in the border then either, the grid is uniquely completable or all multiple completions of the grid are equivalent. An example of a Form 3 unavoidable set of size 12 is given in Figure 3.9 and hence the smallest cardinality of a Form 3 unavoidable set is 12. □

3.4.3.2 Form 4 Unavoidable Sets on Minimum PSCMS of Order 5

Definition 3.51. *Consider a partial PSCMS of order 5,* $\mathbf{H}_5$*, such that all empty cell tuples form a single unavoidable set and are located within the outer border. For the empty cell tuples to form an unavoidable set of Form 4, denoted* $D_5^{\mathbf{H},4}$*, then all completions use the same subset of* $\mathbb{P}'$ *for each of the multiple equivalent and non-equivalent completions in* $\mathbb{A}_5^{\mathbf{H}}$*.*

From Definition 3.51, the multiple completions in $\mathbb{A}_5^{\mathbf{H}}$ will always be a mix of equivalent and non-equivalent completions, but the addition of any triple always results in a partial grid that has multiple equivalent completions. Figure 3.11(a) shows a minimal unavoidable set of Form 4 in a minimum PSCMS of order 5. Like Form 1 and Form 2, Form 4 unavoidable sets are comprised of cells entirely in the border of order 5, and contain the minimum number of empty cells in a border for there to be multiple completions such that at least two are non-equivalent and the list of primes is fixed. Example 3.52 illustrates a partial PSCMS, $\mathbf{H}_5$, multiply completable to minimum PSCMS of order 5, and its two non-equivalent completions. The multiple equivalent completions are not given.

Example 3.52. *Figure 3.11 shows a partial PSCMS of order 5 with a Form 4 unavoidable set of size 8 in cells* $(1,2),(1,3),(2,1),(2,5),(4,1),(4,5),(5,2),(5,3)$ *and the two non-equivalent completions.*

101			449	431
	461	23	269	
353	59	251	443	149
	233	479	41	
71			53	401

(a) Minimal Form 4 Unavoidable Set in a Minimum PSCMS of order 5

101	11	263	449	431
311	461	23	269	191
353	59	251	443	149
419	233	479	41	83
71	491	239	53	401

(b) First Completion of (a)

101	83	191	449	431
239	461	23	269	263
353	59	251	443	149
491	233	479	41	11
71	419	311	53	401

(c) Second Completion of (a)

Figure 3.11: *Minimal Form 4 Unavoidable Set of Size 8 in a Minimum PSCMS of Order 5 with its Two Non-Equivalent Completions*

The completions shown in Figures 3.11(b) and 3.11(c) are non-equivalent and use the same subset of $\mathbb{P}'$. Both completions are minimum PSCMS of order 5, the first completion is of type 1F using complement sets of primes F3 and F4 from Table 2.4 and the second completion is of type 1F using complement sets of primes F5 and F6 from Table 2.4. The addition of any triple to the partial PSCMS in Figure 3.11(a) determines multiple equivalent completions, and hence the empty cells describe an improper unavoidable set.

Lemma 3.53. *An unavoidable set of Form 4 is an improper unavoidable set.*

Proof. A Form 4 unavoidable set satisfies Definition 3.35 and hence condition (2) of Definition 3.32. □

Example 3.54 repeats the partial PSCMS from Figure 3.11(a) and illustrates the outcome of the addition of a triple, producing a partial PSCMS that is completable to two equivalent grids. Hence the unavoidable set is improper.

Example 3.54. *Figure 3.12(a) shows a partial PSCMS, completable to a minimum PSCMS of order 5, with a Form 4 unavoidable set of size 8. Figure 3.12(b) shows, in yellow, the triple (1,2,11) added to (a) and the two equivalent completions are given in Figures 3.12(c) and 3.12(d).*

101			449	431
	461	23	269	
353	59	251	443	149
	233	479	41	
71			53	401

(a) Multiply Completable Partial PSCMS with a Form 4 Improper Unavoidable Set

101	11		449	431
	461	23	269	
353	59	251	443	149
	233	479	41	
71			53	401

(b) Placing the Triple (1,2,11) into (a)

101	11	263	449	431
311	461	23	269	191
353	59	251	443	149
419	233	479	41	83
71	491	239	53	401

(c) First Completion of (b)

101	11	263	449	431
419	461	23	269	83
353	59	251	443	149
311	233	479	41	191
71	491	239	53	401

(d) Second Completion of (b) Equivalent to (c)

Figure 3.12: *Demonstration of Placing a Triple in a Form 4 Unavoidable Set*

Theorem 3.55. *If a partial PSCMS,* $\mathbf{H}_5$*, completable to a minimum PSCMS, contains an unavoidable set of Form 4,* $D_5^{\mathbf{H},4}$*, then* $|D_5^{\mathbf{H},4}| \geq 8$*, and partial PSCMS of order 5 exist that contain such unavoidable sets of size 8.*

Proof. Consider first a partial PSCMS, completable to a minimum PSCMS of order 5, with fewer than six empty cells. If the grid has four empty cells then either it is uniquely completable, and hence the cells do not form an unavoidable set, or it has a Form 1 unavoidable set. Any fewer than four cells can never be an unavoidable set as the grid is always uniquely completable. If the partial PSCMS had exactly six empty cells, then it is immediately uniquely completable, or completable only to equivalent grids, as the subset of $\mathbb{P}'$ is fixed. An example of an unavoidable set, $D_5^{\mathbf{H},4}$, for which $|D_5^{\mathbf{H},4}| = 8$ is given in Figure 3.11. □

3.4.3.3 Form 5 Unavoidable Sets on Minimum PSCMS of Order 5

Definition 3.56. *Consider a partial PSCMS of order 5,* $\mathbf{H}_5$*, such that all empty cell tuples form a single unavoidable set and are located in both the outer border and the subsquare of order 3. For the empty cell tuples to form an unavoidable set of Form 5, denoted* $D_5^{\mathbf{H},5}$*, then all completions use the same subset of* $\mathbb{P}'$*, there are completions with each of the two subsquares of order 3 present, and all completions having the same subsquare are equivalent.*

All grids with Form 5 unavoidable sets have completions using both subsquares, and trivially all pairs of completions using different subsquares are non-equivalent. As with Form 4, the multiple completions in $\mathbb{A}_5^{\mathbf{H}}$ will always be a mix of equivalent and non-equivalent completions, but the addition of any triple always results in a partial grid that has multiple equivalent completions.

Figure 3.13(a) shows a minimal unavoidable set of Form 5 in a minimum PSCMS of order 5. Example 3.57 illustrates a partial PSCMS, $\mathbf{H}_5$, multiply completable to minimum PSCMS of order 5, and its two non-equivalent completions. The multiple equivalent completions are not given.

Example 3.57. *Figure 3.13 shows a partial PSCMS of order 5 with a Form 5 unavoidable set of size 14 in cells* $(1,2), (1,4), (2,2), (2,3), (2,4), (3,1), (3,5), (4,1), (4,2), (4,3),$ $(4,4), (4,5), (5,2), (5,4)$ *and the two non-equivalent completions.*

311		53		401
113				389
	59	251	443	
101		449		191

(a) Minimal Form 5 Unavoidable Set in a Minimum PSCMS of order 5

311	71	53	419	401
113	461	23	269	389
239	59	251	443	263
491	233	479	41	11
101	431	449	83	191

(b) First Completion of (a)

311	11	53	479	401
113	431	83	239	389
269	59	251	443	233
461	263	419	71	41
101	491	449	23	191

(c) Second Completion of (a)

Figure 3.13: *Minimal Form 5 Unavoidable Set of Size 14 in a Minimum PSCMS of order 5 with its Two Non-Equivalent Completions*

The completions shown in Figures 3.13(b) and 3.13(c) are non-equivalent and use the same subset of $\mathbb{P}'$. Both completions are minimum PSCMS of order 5, the first completion is of type 1A using complement sets of primes A1 and A3 from Table 2.2 and the second completion is of type 2C using complement sets of primes C1 and C2 from Table 2.4. The addition of any triple to the partial PSCMS in Figure 3.13(a) determines multiple equivalent completions, and hence the empty cells describe an improper unavoidable set.

A further, detailed analysis of minimal Form 5 unavoidable sets in the minimum PSCMS of order 5 from Chapter 2 is given later in the section. It shall be determined, given a partial PSCMS, completable to a minimum PSCMS of order 5, with a Form 5 unavoidable set, that there are two non-equivalent completions, one of which is a grid of type 1 and the other is a grid of type 2. This is discussed in Theorem 3.64.

Lemma 3.58. *An unavoidable set of Form 5 is an improper unavoidable set.*

Proof. A Form 5 unavoidable set satisfies Definition 3.35 and hence condition (2) of Definition 3.32. □

Example 3.59 repeats the partial PSCMS from Figure 3.13(a) and illustrates the outcome of the addition of a triple, producing a partial PSCMS that is completable to four equivalent grids. Hence the unavoidable set is improper.

Example 3.59. *Figure 3.14(a) shows a partial PSCMS, completable to a minimum PSCMS of order 5, with a Form 5 unavoidable set of size 14. Figure 3.14(b) shows, in yellow, the triple (1,2,71) added to (a) and the four equivalent completions are given in Figures 3.14(c), 3.14(d), 3.14(e) and 3.14(f).*

311		53		401
113				389
	59	251	443	
101		449		191

(a) Multiply Completable Partial PSCMS with a Form 5 Improper Unavoidable Set

311	71	53		401
113				389
	59	251	443	
101		449		191

(b) Placing the Triple (1,2,71) into (a)

311	71	53	419	401
113	461	23	269	389
239	59	251	443	263
491	233	479	41	11
101	431	449	83	191

(c) First Completion of (b)

311	71	53	419	401
113	461	23	269	389
491	59	251	443	11
239	233	479	41	263
101	431	449	83	191

(d) Second Completion of (b) Equivalent to (c)

311	71	53	419	401
113	233	479	41	389
239	59	251	443	263
491	461	23	269	11
101	431	449	83	191

(e) Third Completion of (b) Equivalent to (c) and (d)

311	71	53	419	401
113	233	479	41	389
491	59	251	443	11
239	461	23	269	263
101	431	449	83	191

(f) Fourth Completion of (b) Equivalent to (c), (d) and (e)

Figure 3.14: *Demonstration of Placing a Triple in a Form 5 Unavoidable set*

Lemma 3.60. *If a partial PSCMS, $\mathbf{H}_5$, completable to a minimum PSCMS, contains an unavoidable set of Form 5, $D_5^{\mathbf{H},5}$, then $|D_5^{\mathbf{H},5}| \geq 12$.*

Proof. There exist two unique subsquares of order 3 for the minimum PSCMS of order 5, and these subsquares consist of the centre cell value M and four pairs of complement primes, one of which is common to both subsquares. Hence, the existence of completions using both subsquares requires that three paired cells (six cells) are empty in the centre subsquare and three paired cells (six cells) in the border are empty. Therefore, at least twelve cells are empty. If fewer cells are empty in the border, since all completions use the same subset of $\mathbb{P}'$ the partial grid is completable only to equivalent grids. $\square$

If three pairs of cells are empty in the subsquare of order 3 and three pairs are empty in the border, there are five distinctly different patterns in which these empty cells could be arranged, shown in Figure 3.15 where the cells with letters are the empty ones.

	g	h	i	
	a	b	c	
	d	e	f	
	j	k	l	

(a) Case 1

	g			
i	a	b	c	k
j				l
	d	e	f	
	h			

(b) Case 2

g				
h	a	b	c	j
i				k
	d	e	f	
				l

(c) Case 3

g	h			
i	a	b	c	j
	d	e	f	
	k			l

(d) Case 4

i	k			g
	a	b	c	
	d	e	f	
h	l			j

(e) Case 5

Figure 3.15: *Patterns of Twelve Empty Cells in Grids of Order 5 Where Six of the Empty Cells are in the Order 3 Subsquare*

It will now be shown that none of these patterns of twelve empty cells can form a Form 5 unavoidable set.

Lemma 3.61. *There is no unavoidable set of Form 5, $D_5^{\mathbf{H},5}$, such that $|D_5^{\mathbf{H},5}| = 12$.*

Proof. It can easily be seen that the patterns relating to Cases 2 and 3 in Figure 3.15 are always completable only to equivalent grids since they contain a forced completable cell in the border. Now considering Cases 3, 4 and 5, for every pattern of empty cells, where $M = 251$, either the partial grid is completable only to equivalent grids or the multiple completions use different lists of primes. Hence there are no Form 5 unavoidable sets in any minimum PSCMS of order 5 having only twelve empty cells. □

Theorem 3.62. *If a partial PSCMS, $\mathbf{H}_5$, completable to a minimum PSCMS, contains an unavoidable set of Form 5, $D_5^{\mathbf{H},5}$, then $|D_5^{\mathbf{H},5}| \geq 14$, and partial PSCMS of order 5 exist that contain such unavoidable sets of size 14.*

Proof. From Lemmas 3.60 and 3.61, $|D_5^{\mathbf{H},5}| > 12$. Since an unavoidable set consists of paired cells, then $|D_5^{\mathbf{H},5}| \geq 14$, and an example of an unavoidable set of Form 5 consisting of 14 cells is given in Example 3.57. □

If a partial PSCMS, completable to a minimum PSCMS of order 5 (given in Chapter 2), contains a single Form 5 unavoidable set, then it has completions including both subsquares of order 3. The requirement to use the same subset of $\mathbb{P}'$ to complete the grid restricts the completions to two non-equivalent grids, one of which is type 1 and one of which is type 2. This establishes a relationship between pairs of specific subtypes. The following Lemma establishes which subtypes cannot be in such a pairing, and Theorem 3.64 specifies all valid pairings.

Lemma 3.63. *No unavoidable set of Form 5 exists in minimum PSCMS of order 5 of types 1B, 2A, 2G, 2H or 2F.*

Proof. For a minimum PSCMS of order 5, $\mathbb{P}' = \{11, 23, 41, 53, 59, 71, 83, 101, 113, 149, 191, 233, 239, 263, 269, 311, 353, 389, 401, 419, 431, 443, 449, 461, 479, 491\}$.

For type 1B the omitted pair is $(71, 431)$, which occurs in the subsquare of type 2. Hence an unavoidable set of Form 5 does not exist in a PSCMS of type 1B.

For types 2A, 2G and 2H the omitted pairs are $(233, 269)$, $(41, 461)$ and $(23, 479)$ respectively, which all occur in the subsquare of type 1. Hence an unavoidable set of Form 5 does not exist in PSCMS of types 2A, 2G and 2H.

For type 2F the omitted pair is $(101, 401)$, which is not found in the type 1 subsquare, however it is in every type 1 border of order 5. Hence an unavoidable set of Form 5 does not exist in a PSCMS of type 2F. □

Theorem 3.64. *Form 5 unavoidable sets exist in minimum PSCMS of order 5 of types 1A, 1C, 1D, 1E, 1F, 2B, 2C, 2D, 2E, 2I.*

Proof. For a minimum PSCMS of order 5, $\mathbb{P}' = \{11, 23, 41, 53, 59, 71, 83, 101, 113, 149, 191, 233, 239, 263, 269, 311, 353, 389, 401, 419, 431, 443, 449, 461, 479, 491\}$.

Consider a partial PSCMS, $\mathbf{H}_5$, completable to a minimum PSCMS, with an unavoidable set of Form 5. If $\mathbf{H}_5$ is completable using $\mathbb{P}'$ by omitting the pair $(149, 353)$ then $\mathbf{H}_5$ is completable to either type 1A or type 2C, and $|\mathbb{A}_5^{\mathbf{H}}| > 2$, exactly two of which are non-equivalent. If the omitted pair from $\mathbb{P}'$ is $(53, 449)$ then $\mathbf{H}_5$ is completable to either type 1C or type 2D, and $|\mathbb{A}_5^{\mathbf{H}}| > 2$, exactly two of which are non-equivalent. If the omitted pair from $\mathbb{P}'$ is $(11, 491)$ then $\mathbf{H}_5$ is completable to either type 1D or type 2I, and $|\mathbb{A}_5^{\mathbf{H}}| > 2$, exactly two of which are non-equivalent. If the omitted pair from $\mathbb{P}'$ is $(191, 311)$ then $\mathbf{H}_5$ is completable to either type 1E or type 2B, and $|\mathbb{A}_5^{\mathbf{H}}| > 2$, exactly two of which are non-equivalent. If the omitted pair from $\mathbb{P}'$ is $(113, 389)$ then $\mathbf{H}_5$ is completable to either type 1F or type 2E, and $|\mathbb{A}_5^{\mathbf{H}}| > 2$, exactly two of which are non-equivalent. □

3.4.3.4 Summary of Unavoidable Sets on Minimum PSCMS of Order 5

Theorem 3.65. *If a partial PSCMS,* $\mathbf{H}_5$*, completable to a minimum PSCMS, contains an unavoidable set of:*

(1) Form 1, the minimum cardinality of the unavoidable set is 4.

(2) Form 2, the minimum cardinality of the unavoidable set is 6.

(3) Form 3, the minimum cardinality of the unavoidable set is 12.

(4) Form 4, the minimum cardinality of the unavoidable set is 8.

(5) Form 5, the minimum cardinality of the unavoidable set is 14.

Proof. Follows from Theorems 3.40, 3.45, 3.50, 3.55 and 3.62. □

It can be noted that, from Definition 3.15, a forced completable set of a SCMS, $\mathbf{F}_n$, is a set of triples such that the partial SCMS, $\mathbf{H}_n$, so defined is strongly completable. A forced completable set ensures that there are no possible unavoidable sets of any form in the partial completable grid. If this is not the case then the grid is not strongly completable. Hence, the forced completable set includes at least one triple from every possible unavoidable set.

3.5 Conclusion of Partial SCMS

Generalising known concepts for Latin Squares and Sudoku grids, definitions of completability are defined for Magic Squares. The notion that a strongly completable grid is a valid puzzle is introduced as such a grid is completable using logical deductions. The idea of a forced completable set is then defined as the set of triples specifying the non-empty cells of a strongly completable partial SCMS, before defining a critical set of a SCMS as a set of triples which specify the non-empty cells of a uniquely completable SCMS. The latter differs from a forced completable set as grids exist that are weakly uniquely completable. Comparisons are then made between the cardinality of forced completable sets and critical sets and the minimal cardinality in each case is found in a minimum PSCMS of order 5 from Chapter 2. This is followed by the introduction of unavoidable sets, again a concept from Latin Squares literature, however these differ because of the added constraints of a SCMS, namely the grid being strictly concentric and all values being distinct. Due to the complex nature of these unavoidable sets they are split into proper/improper categories and subcategorised into forms based on cell patterns. The cardinality of each form of unavoidable set is investigated for the minimum PSCMS of order 5 (given in Chapter 2).

Chapter 4

Prime Strictly Concentric Magic Squares of Higher Order

Having provided the foundational work concerning properties of SCMS and PSCMS, and the enumeration of minimum PSCMS of order 5, it is now possible to explore PSCMS of higher order. Recall that no formal results on PSCMS of higher order are present in the literature. In this Chapter, fundamental work is presented on PSCMS of higher order. Section 4.1 provides an algorithm for the construction of minimum PSCMS of order 7, and a full novel enumeration with some analysis. Then a general algorithm for minimum PSCMS of order n, n odd, is given with a discussion of why enumeration rapidly increases in complexity with increasing order. Section 4.2 explores PSCMS of even order and new definitions are given where they necessarily differ from the odd order. An algorithm for the construction of minimum PSCMS of order 6 is given, followed by a general algorithm for minimum PSCMS of order n, n even.

4.1 General PSCMS of Odd Order

Using the definitions in Chapters 1 and 2, recall the following information for PSCMS of odd order. From Theorem 1.12, the number of border pairs of a SCMS of order n, n odd, is $\frac{n^2-1}{2}$

and so in order to form a PSCMS of order n, n odd, there must exist a prime M and $\frac{n^2-1}{2}$ pairs of complement primes summing to $2M$. Then from Theorem 2.1, the Magic Square has magic constant $S_n = nM$.

4.1.1 Minimum PSCMS of Order 7

4.1.1.1 Introduction

A SCMS of order 7 comprises a centre cell value, M, and twenty-four distinct pairs of values summing to $2M$, four of which surround the centre cell forming the subsquare of order 3, eight of which form the border of the subsquare of order 5, and twelve of which form the outer border of order 7. One such order 7 border example is shown in Figure 4.1(a) and one such order 5 border example is shown in Figure 4.1(b) with one border pair in each shaded in grey. By Lemma 2.1, a SCMS of order 7 has magic constant $S_7 = 7M$.

Recall, from Definition 1.17, a minimum PSCMS has minimum M value. From Lemma 2.20, the minimum PSCMS of order 7 cannot have a minimum PSCMS of order 5 as a subsquare, and nor can it have a minimum PSCMS of order 3 as a subsquare due to the requirement on M. A minimum PSCMS of order 7 is given in [25] to have $S_7 = 4,487$ with centre cell value $M = 641$, and border pairs summing to $2M = 1,282$. The magic constant for the subsquare of order 5 is $5M = 3,205$ and the magic constant for the subsquare of order 3 is $3M = 1,923$, as is evident in Figure 4.1(c). It is proved by the current author in Lemma 4.1 that the minimum PSCMS of order 7 has $M = 641$ and hence from Lemma 2.1, $S_7 = 4,487$ and therefore the placement of primes in Figure 4.1 (generated by [42]) form a minimum PSCMS of order 7.

1151	311	461	521	881	929	233
59						1223
251						1031
269			641			1013
599						683
1109						173
1049	971	821	761	401	353	131

(a) Border of a Minimum PSCMS of Order 7 With Centre Cell Value 641 with one Border Pair Shaded in Grey

	1277	263	773	839	53	
	89				1193	
	191		641		1091	
	419				863	
	1229	1019	509	443	5	

(b) Border of Order 5 of a Minimum PSCMS of Order 7 With Centre Cell Value 641 with one Border Pair Shaded in Grey

		1181	179	563		
		23	641	1259		
		719	1103	101		

(c) Magic Subsquare of Order 3 for a Minimum PSCMS of Order 7

Figure 4.1: *Border and Magic Subsquares of Order 3 and Order 5 for a Minimum PSCMS of Order 7*

Recall that $\mathbb{P}$ denotes the set of all primes numbers and $\mathbb{P}'$ any subset of $\mathbb{P}$.

Lemma 4.1. *The minimum PSCMS of order 7 has centre cell value of 641 and magic constant of 4,487.*

Proof. Assume for contradiction that the centre cell value, M, is less than 641. Let $\mathbb{P}$ be the set of all prime numbers. To construct a PSCMS for which $M < 641$ and prime, there must exist at least twenty-four distinct pairs of complement primes $x_i, \bar{x}_i \in \mathbb{P}$ such that $x_i + \bar{x}_i = 2M$, where $x_i, \bar{x}_i$ are the values in paired cells. These pairs of complement primes form $\mathbb{P}'$. Four of these pairs form the border pairs ($\mathbb{B}_3$) of the subsquare of order 3, eight form the border pairs ($\mathbb{B}_5$) of the subsquare of order 5, and twelve form the outer border pairs ($\mathbb{B}_7$) of the PSCMS of order 7. There does not exist a prime $M < 641$ for which there are twenty-four such pairs of complement primes. It is known that a PSCMS of order 7 with centre cell value 641 exists, and one example is given in Figure 4.2. Hence, any PSCMS of order 7 with centre cell value 641 and magic constant $S_7 = 4,487$ is a minimum PSCMS. □

1151	311	461	521	881	929	233
59	1277	263	773	839	53	1223
251	89	1181	179	563	1193	1031
269	191	23	641	1259	1091	1013
599	419	719	1103	101	863	683
1109	1229	1019	509	443	5	173
1049	971	821	761	401	353	131

Figure 4.2: *A Minimum PSCMS of Order 7*

4.1.1.2 Construction of Minimum PSCMS of Order 7

Having established M for all minimum PSCMS of order 7, this section details the construction of such a square.

Lemma 4.2. *There are six possible non-equivalent magic subsquares of order 3 for the minimum PSCMS of order 7.*

Proof. Given the centre cell value $M = 641$, it can be easily determined that there are twenty-four pairs of complement primes satisfying Lemma 2.2, with squares given in Figure 4.3. □

To enumerate the minimum PSCMS of order 7, either the PSCMS of order 5 with $M = 641$ can be generated to explore whether a border can be added to form a grid of order 7, or the grids of order 7 can be formed directly around the grids of order 3 from Figure 4.3.

Consider first the generation of the PSCMS of order 5 with $M = 641$. The current author requested that a computationally efficient program be written to employ Algorithms 2 and 3 [42]. Results given in previous chapters were calculated by the current author, and checked using this implementation of algorithms devised by the author. In this chapter, due to the number of the PSCMS, this implementation of the algorithms was used to provide the results. The program generated 162436 PSCMS of order 5. Two illustrative examples are given in Figure 4.4, the first (Figure 4.4(a)) containing magic subsquare 1 of order 3 shown in Figure 4.3(a) and the second (Figure 4.4(b)) containing magic subsquare 6 of order 3 shown in Figure 4.3(f). Figure 4.4 also shows two examples of PSCMS of order 7 with the first (Figure 4.4(c)) containing the grid in Figure 4.4(a) as the magic subsquare of order 5 and the second (Figure 4.4(d)) containing the grid in Figure 4.4(b) as the magic subsquare of order 5. It is not known *a priori* which PSCMS of order 5 are subsquares of valid PSCMS of order 7 and, given the large number of generated PSCMS of order 5, it is considered beyond the scope of this thesis to explore them.

Consider instead forming the PSCMS of order 7 around one of the six grids of order 3. The existing program [42] enables manual entry of a subsquare of order i for $i \geq 1$ and odd, into an empty grid of order n and (if computationally feasible) the PSCMS of order n are generated. While the entry of 162436 grids of order 5 was infeasible as an approach to generate the grids of order 7, having only six grids of order 3 to enter made this a feasible approach.

1181	179	563
23	641	1259
719	1103	101

(a) Magic Subsquare 1

1151	173	599
89	641	1193
683	1109	131

(b) Magic Subsquare 2

971	599	353
23	641	1259
929	683	311

(c) Magic Subsquare 3

971	443	509
179	641	1103
773	839	311

(d) Magic Subsquare 4

971	353	599
269	641	1013
683	929	311

(e) Magic Subsquare 5

761	599	563
443	641	839
719	683	521

(f) Magic Subsquare 6

Figure 4.3: *The Six Non-Equivalent Magic Subsquares of Order 3 for a Minimum PSCMS of Order 7*

761	173	443	1229	599
1193	1181	179	563	89
509	23	641	1259	773
59	719	1103	101	1223
683	1109	839	53	521

(a) Magic Subsquare of Order 5 with Subsquare 1 (Figure 4.3(a)) of Order 3

821	233	419	1223	509
1277	761	599	563	5
311	443	641	839	971
23	719	683	521	1259
773	1049	863	59	461

(b) Magic Subsquare of Order 5 with Subsquare 6 (Figure 4.3(f)) of Order 3

881	251	269	419	929	1277	461
1091	761	173	443	1229	599	191
1019	1193	1181	179	563	89	263
311	509	23	641	1259	773	971
233	59	719	1103	101	1223	1049
131	683	1109	839	53	521	1151
821	1031	1013	863	353	5	401

(c) PSCMS of Order 7 with (a) as the Subsquare of Order 5

1019	251	269	353	1091	1103	401
1193	821	233	419	1223	509	89
1109	1277	761	599	563	5	173
131	311	443	641	839	971	1151
101	23	719	683	521	1259	1181
53	773	1049	863	59	461	1229
881	1031	1013	929	191	179	263

(d) PSCMS of Order 7 with (b) as the Subsquare of Order 5

Figure 4.4: *Two Examples of Magic Subsquares of Order 5 Generated by [42] and Completed to Minimum PSCMS of Order 7*

All minimum PSCMS of order 7 consist of a centre cell value and twenty-four pairs of complement primes formed from the following set of 48 prime numbers: $\mathbb{P}' = \{5, 23, 53, 59, 89, 101, 131, 173, 179, 191, 233, 251, 263, 269, 311, 353, 401, 419, 443, 461, 509, 521, 563, 599, 683, 719, 761, 773, 821, 839, 863, 881, 929, 971, 1013, 1019, 1031, 1049, 1091, 1103, 1109, 1151, 1181, 1193, 1223, 1229, 1259, 1277\}$. All minimum PSCMS of order 7 include all the primes in this set as there are exactly the number required to fill a grid of order 7.

Algorithm 3 uses Algorithm 2 to form a subsquare of order 5, with chosen M and $\mathbb{P}'$, before using a similar process to construct a PSCMS of order 7 around the given subsquare, if one exists.

Algorithm 3 *Algorithm to form a PSCMS of order 7*

Begin
Input M and form $\mathbb{P}'$, the set of all primes that form pairs summing to $2M$.
repeat
    Construct a PSCMS of order 5 with chosen M and $\mathbb{P}'$ using Algorithm 2.
    Place the magic subsquare of order 5, with centre cell value M, into the centre of an empty grid of order 7.
    Form a set Q of the primes from $\mathbb{P}'$ not used in the subsquare.
    repeat
        Take a set of seven distinct non-paired primes from Q that sum to $7M$, to form a set S, and their complements to form a set $\bar{S}$.
        Take a set of five distinct non-paired primes from Q to form a set T, and their complements to form a set $\bar{T}$.
        repeat
            Take an element x of S, and an element y of $\bar{S}$ that is not paired with x.
            Let X be the sum of x, y and the elements of T.
        until $X = 7M$, or no further combinations of x, y are possible.
    until $X = 7M$, or no further combinations of S are possible.
until $X = 7M$, or no further PSCMS of order 5 can be generated using Algorithm 2.
if $X = 7M$ **then**
    Begin
    Place x in $(1,1)$, y in $(1,7)$ removing them from S and $\bar{S}$, and place their complements in the paired cells, removing them from $\bar{S}$ and S.
    Place the elements of T in $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$ in any order, and place their complements from $\bar{T}$ in the paired cells.
    Place the remaining elements of S in $(2,1)$, $(3,1)$, $(4,1)$, $(5,1)$, $(6,1)$ in any order and place their complements from $\bar{S}$ in the paired cells.
    End
else
    No PSCMS exists for the placed centre subsquare.
end if
End

Theorem 4.3. *A minimum PSCMS of order 7 with centre cell value 641 is always formed using Algorithm 3 when magic subsquare 1,2,3,4,5 or 6 (shown in Figure 4.3) is placed in the centre of the grid.*

Proof. For $M = 641$, the set of complement primes (primes summing to 1282) $\mathbb{P}' = \{5, 23, 53, 59, 89, 101, 131, 173, 179, 191, 233, 251, 263, 269, 311, 353, 401, 419, 443, 461, 509, 521, 563, 599, 683, 719, 761, 773, 821, 839, 863, 881, 929, 971, 1013, 1019, 1031, 1049, 1091, 1103, 1109, 1151, 1181, 1193, 1223, 1229, 1259, 1277\}$. Recall from Definition 2.3 that all border pairs are pairs of complement primes. Twenty-four of these primes must be used in the subsquare of order 5 and removed from $\mathbb{P}'$ to form a set Q.

In each case the remaining twenty-four primes are all used in the border of order 7. Using a valid subsquare of order 5, there are seven distinct pairs of complement primes in Q which satisfy conditions (1) and (3) below. From the remaining primes in Q it is possible to check that there are always five more pairs of complement primes that can be chosen to satisfy conditions (2) and (4) below.

(1) $a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} + a_{17} = 4487$

(2) $a_{11} + a_{21} + a_{31} + a_{41} + a_{51} + a_{61} + a_{71} = 4487$

(3) $a_{71} + a_{72} + a_{73} + a_{74} + a_{75} + a_{76} + a_{77} = 4487$

(4) $a_{17} + a_{27} + a_{37} + a_{47} + a_{57} + a_{67} + a_{77} = 4487$.

The primes are placed in the manner specified in Algorithm 3 (with the subsquare of order 5 having been determined in line 4, through a call to Algorithm 2, the paired values being determined in lines 7 to 15, and the border of the PSCMS of order 7 then being filled by lines 18 to 20). A minimum PSCMS is thereby formed. □

4.1.1.3 Enumeration of Minimum PSCMS of Order 7

The program based on Algorithm 3 [42], and Table 1.1 are used to determine the number of minimum PSCMS of order 7. Firstly, each of the six magic subsquares of order 3 with $M = 641$, shown in Figure 4.3, are placed into an empty grid of order 5 and the number of PSCMS of order 5 with each magic subsquare of order 3 is:

(1) 28594 with magic subsquare 1

(2) 31462 with magic subsquare 2

(3) 28591 with magic subsquare 3

(4) 27001 with magic subsquare 4

(5) 26136 with magic subsquare 5

(6) 20652 with magic subsquare 6

These six values give a total of 162436 non-equivalent Magic Squares of order 5 which could potentially be subsquares of a minimum PSCMS of order 7.

As discussed in Section 4.1.1.2, the subsquares of order 3 are used to generate the grids of order 7 rather than the subsquares of order 5. Therefore, each of the six magic subsquares of order 3 with $M = 641$ are placed into an empty grid of order 7 and the number of PSCMS of order 7 with each magic subsquare of order 3 is:

(1) 2255115 with magic subsquare 1

(2) 2383510 with magic subsquare 2

(3) 2008431 with magic subsquare 3

(4) 1782479 with magic subsquare 4

(5) 1777400 with magic subsquare 5

(6) 1261292 with magic subsquare 6

This gives a total of 11,468,227 non-equivalent minimum PSCMS of order 7.

Lemma 4.4. *There are 11,468,227 non-equivalent minimum PSCMS of order 7.*

Proof. Using centre cell value 641 and the list of primes $\mathbb{P}'$, there are six Magic Squares of order 3 and each of these are magic subsquares of minimum PSCMS of order 7. The total of the generated squares with each of the six subsquares of order 3 is 11,468,227. □

Theorem 4.5. *There are 3,043,905,984,921,600 minimum PSCMS of order 7 and these have magic constant 4,487.*

Proof. This proof follows directly from Lemma 4.4 and Equation 1.1 at each border. □

4.1.1.4 Conclusion of Minimum PSCMS of Order 7

It has been determined that there exist 11,468,227 non-equivalent minimum PSCMS of order 7, and these have magic constant 4,487. Hence, using the permutations from Table 1.1 there exist 3,043,905,984,921,600 minimum PSCMS of order 7. Having enumerated the minimum PSCMS of both order 5 and 7, it can be conjectured that similar techniques could be used to enumerate higher order PSCMS of order n where n is fixed and odd. Considering the increase from the number of PSCMS of order 5 to the number of PSCMS of order 7, substantial computing power is currently necessary in order to gain exact results for any higher order grid. The minimum centre cell values of PSCMS of odd order 5 to 19 have previously been determined [25] and using these values the number of Magic Squares of order 3 with the given centre cell value are generated by the current author using [42].

Order	Centre [25]	No. Subsquares of Order 3
5	251	2
7	641	6
9	1361	9
11	2411	23
13	3803	38
15	4973	74
17	7541	75
19	10061	119

Table 4.1: *Number of Subsquares of Order 3 for Minimum PSCMS*

As the number of potential subsquares of order 3 increases, the number of non-equivalent Magic Squares of order 5 increases. For the minimum PSCMS of order 5 there are 35 non-equivalent squares, for the minimum PSCMS of order 7 there are 162436 Magic Squares of order 5 with $M = 641$ and for the minimum PSCMS of order 9 there are 13456126 Magic Squares of order 5 with $M = 1361$ generated using [42].

4.1.2 Construction of PSCMS of Odd Order

Algorithm 4 uses Algorithm 1 to form a subsquare of order 3, with chosen M and $\mathbb{P}'$, before using a similar process to construct a PSCMS of order n, n odd, around the given subsquare, if one exists.

Algorithm 4 *Algorithm to form a PSCMS of order n, n odd*

Begin
Input M and form $\mathbb{P}'$, the set of all primes that form pairs summing to $2M$.
repeat
  Construct a PSCMS of order 3 with chosen M and $\mathbb{P}'$ using Algorithm 1.
  Place the magic subsquare of order 3, with centre cell value M, into the centre of an empty grid of order n.
  Form a set Q of the primes from $\mathbb{P}'$ not used in the subsquare.
  Let $i = 3$.
  repeat
    repeat
      Let $i = i + 2$.
      Take a set of i distinct non-paired primes from Q that sum to iM, to form a set S, and their complements to form a set $\bar{S}$.
      Take a set of $i - 2$ distinct non-paired primes from Q to form a set T, and their complements to form a set $\bar{T}$.
      repeat
        Take an element x of S, and an element y of $\bar{S}$ that is not paired with x.
        Let X be the sum of x, y and the elements of T.
      until $X = iM$, or no further combinations of x, y are possible.
    until $X = iM$, or no further combinations of S are possible.
  until $X = iM$, or no further PSCMS of order 3 can be generated using Algorithm 1.
  if $X = iM$ **then**
    Begin
    Place x in $(1, 1)$, y in $(1, i)$ removing them from S and $\bar{S}$, and place their complements in the paired cells, removing them from $\bar{S}$ and S.
    Place the elements of T in $(1, 2)$,..., $(1, i - 1)$ in any order, and place their complements from $\bar{T}$ in the paired cells.
    Place the remaining elements of S in $(2, 1)$,..., $(i - 1, 1)$ in any order, and place their complements from $\bar{S}$ in the paired cells.
    End
  else
    No PSCMS exists for the placed centre subsquare.
  end if
until $i = n$
End

For a PSCMS of order n, $n \geq 5$ the number of magic subsquares of order 3 depends on M and therefore, it is not known in general how many are valid for a given PSCMS. In Section 2.2.2 there are two valid subsquares of order 3 for the minimum PSCMS of order 5 and in Section 4.1.1.2 there are six valid subsquares of order 3 for the minimum PSCMS of order 7.

Theorem 4.6. *A PSCMS of order n, n odd, with centre cell value M is always formed using Algorithm 4 when every subsquare of order $n-2i$, $i=1,\ldots,\dfrac{(n-3)}{2}$, is a valid PSCMS.*

Proof. For a given M, the complement primes (primes summing to $2M$) form a set $\mathbb{P}'$. The set must have cardinality $|\mathbb{P}'| \geq n^2-1$. Now, n^2-4n-5 of these primes must be used in the subsquare and removed from $\mathbb{P}'$ to form a set Q. Then $4(n-1)$ primes from Q are used in the border of order n. Given that the subsquare of order $n-2$ is a valid PSCMS, there are n distinct pairs of complement primes in Q which satisfy conditions (1) and (3) below. From the remaining primes in Q it is possible to check that there are always $n-2$ more pairs of complement primes that can be chosen to satisfy conditions (2) and (4) below.

(1) $a_{11}+a_{12}+\cdots+a_{1n}=nM$

(2) $a_{11}+a_{21}+\cdots+a_{n1}=nM$

(3) $a_{n1}+a_{n2}+\cdots+a_{nn}=nM$

(4) $a_{1n}+a_{2n}+\cdots+a_{nn}=nM$

The primes are placed in the manner specified in Algorithm 4 (with the subsquare of order 3 having been determined in line 4, through a call to Algorithm 1, the paired values being determined in lines 9 to 18, and the border of the PSCMS of order n then being filled by lines 21 to 23). A PSCMS is thereby formed. □

4.2 General PSCMS of Even Order

4.2.1 Introduction

This section provides an introduction to PSCMS of even order. Firstly definitions are given where they necessarily differ from the odd order and then an algorithm for the construction of minimum PSCMS of order 6 is given.

Lemma 4.7. *For n even, the subsquares of a Magic Square of order n are of order $m = n-2i$, $i = 1, \ldots, \frac{n-2}{2}$. The smallest such subsquare is of order 2.*

Proof. Proof follows immediately from Definition 1.3. □

As shown in Section 1.1 the squares of order 2 do not conform to the properties of a Magic Square and hence the smallest magic subsquare of a PSCMS of order n, n even, is of order 4. In order to conceptualise SCMS of even order in a similar way to that of odd order, the subsquare of order 2 is here considered to be similar to a centre cell value for odd order as pairs of primes can be placed around the subsquare of order 2 to form a Magic Square of order 4. Hence the Magic Square of order 4 is here considered a trivial Concentric Magic Square.

Definition 4.8. *A Magic Square of order n, $n \geq 6$ and even, is **Strictly Concentric**, denoted a SCMS, if each of its order $m = n - 2i$ subsquares, $i = 1, \ldots, \frac{n-4}{2}$, is a CMS. A subsquare of order 4 is here considered a trivial SCMS. A SCMS containing n^2 distinct primes is denoted a PSCMS.*

Consideration is now given to construction of the subsquares of order 4. There are many subsets of Magic Squares of order 4, and one such subset is associative Magic Squares. Recall Definition 1.5, if all sums of values in pairs of cells symmetric about the centre are equal then the Magic Square is referred to as associative. In order to generalise analysis on SCMS of even order it is important that all subsquares are formed in the same way and that the placement of complement pairs of values is consistent. Therefore this thesis will only consider associative Magic Squares of order 4 as valid subsquares for SCMS of higher even order. Hence, for paired cells in Definition 4.9 it is assumed that the subsquare of order 4 is associative, then once the border of order 6 is added the square is concentric and no longer associative. It is proven that a Magic Square of order n, $n \geq 6$ and even, cannot be both associative and concentric in Lemma 4.18.

Definition 4.9. *For a SCMS of order n, n even, a cell in row i, column j has a* ***paired cell*** *in row $\bar{i}$, column $\bar{j}$, such that*

$$
(\bar{i},\bar{j}) = \begin{cases}
(n-i+1, n-j+1) & i = j,\ i = 1,\dots,n \qquad (1)\\
(n-i+1, i) & j = n-i+1,\ i = 1,\dots,n \qquad (2)\\
(i, n-j+1) & i = 2,\dots,n-1,\ j \neq i,\quad j = 1,\dots,\frac{n-4}{2} \quad \textit{and} \quad j = \frac{n+6}{2},\dots,n\\
 & i+j \leq n \quad \textit{when} \quad i > j \quad \textit{and} \quad i+j \geq n+2 \quad \textit{when} \quad i < j \qquad (3)\\
(n-i+1, j) & j = 2,\dots,n-1,\ j \neq i,\quad i = 1,\dots,\frac{n-4}{2} \quad \textit{and} \quad i = \frac{n+6}{2},\dots,n\\
 & i+j \leq n \quad \textit{when} \quad j > i \quad \textit{and} \quad i+j \geq n+2 \quad \textit{when} \quad j < i \qquad (4)\\
(\frac{n+4}{2}, \frac{n+2}{2}) & i = \frac{n-2}{2},\ j = \frac{n}{2} \qquad (5)\\
(\frac{n+4}{2}, \frac{n}{2}) & i = \frac{n-2}{2},\ j = \frac{n+2}{2} \qquad (6)\\
(\frac{n+2}{2}, \frac{n+4}{2}) & i = \frac{n}{2},\quad j = \frac{n-2}{2} \qquad (7)\\
(\frac{n}{2}, \frac{n+4}{2}) & i = \frac{n+2}{2},\ j = \frac{n-2}{2} \qquad (8)
\end{cases}
$$

Figure 4.5 illustrates the conditions on paired cells given in Definition 4.9 for a SCMS of order 8.

1a	4a	4b	4c	4d	4e	4f	2a
3a	1b	4g	4h	4i	4j	2b	3a
3b	3g	1c	5	6	2c	3g	3b
3c	3h	7	1d	2d	8	3h	3c
3d	3i	8	2d	1d	7	3i	3d
3e	3j	2c	6	5	1c	3j	3e
3f	2b	4g	4h	4i	4j	1b	3f
2a	4a	4b	4c	4d	4e	4f	1a

Figure 4.5: *Illustration of Paired Cells for a PSCMS of Order 8; the Cell Numbers Relate to the Equation Numbers Given in Definition 4.9, Followed by a Letter Denoting Pairings*

Definition 4.10. *A SCMS of order n, n even, and each of its subsquares, has a border which comprises those cells which are adjacent to its respective outer edge. Let B_n be the set of* ***border cells*** *of the SCMS of order n, and B_{n-2i} be the set of border cells of its subsquares of order $n-2i$, $i = 1, \ldots, \frac{n-4}{2}$. B^n denotes the set of all such border cells for an SCMS of order n; $B^n = \cup B_{n-2i}$, $i = 0, \ldots, \frac{n-4}{2}$.*

Definition 4.11. *A* ***border pair*** *$(a_{ij}, a_{\bar{i}\bar{j}})$ is a pair of values placed in cells in B^n for a SCMS of order n, n even, where (i, j) and $(\bar{i}, \bar{j})$ are paired cells. Let $\mathbb{B}_n$ be the set of border pairs of the SCMS of order n, and $\mathbb{B}_{n-2i}$ be the set of border pairs of its subsquares of order $n-2i$, $i = 1, \ldots, \frac{n-4}{2}$. Let $\mathbb{B}^n$ denote the set of all such border pairs for a SCMS of order n; $\mathbb{B}^n = \cup \mathbb{B}_{n-2i}$, $i = 0, \ldots, \frac{n-4}{2}$, $|\mathbb{B}^n| = \frac{|B^n|}{2}$.*

Lemma 4.12. *The number of border pairs, $|\mathbb{B}^n|$, of a SCMS of order n, n even, $n \geq 4$ is $|\mathbb{B}^n| = 2(n-1) + |\mathbb{B}^{n-2}|$ where $|\mathbb{B}^2|$ is taken to be 2.*

Proof. Although the order 2 subsquare of a SCMS is not itself a Magic Square, it is intuitive to say that it has two border pairs, and this aids analysis.
Let $n = 4$, $|\mathbb{B}^4| = 8$ from observation and satisfies the given recurrence. Assume the recurrence is true for some $n = k$, $k > 4$ and even, $|\mathbb{B}^k| = 2(k-1) + |\mathbb{B}^{k-2}|$.
Now consider the case $n = k+2$, for which $2((k+2)-1)$ border pairs are added to $\mathbb{B}^k$, hence $|\mathbb{B}^{k+2}| = 2((k+2)-1) + |\mathbb{B}^k|$. By induction the recurrence holds for any n even. □

From the recurrence the integer sequence obtained is A001105 [39].

Theorem 4.13. *The number of border pairs $|\mathbb{B}^n|$ of a SCMS of order n, n even, $n \geq 4$ is $|\mathbb{B}^n| = \frac{n^2}{2}$.*

Proof. For j even, $j = 2i$ where $i = 1, \ldots, \frac{n}{2}$, the number of pairs for each border of an order j subsquare is $|\mathbb{B}_j| = 2(j-1)$. From the proof of Lemma 4.12, the order n SCMS has $|\mathbb{B}_n| = 2(n-1)$ border pairs. Hence, $|\mathbb{B}_j| = 2(2i-1) = 4i-2$. Hence, $|\mathbb{B}^n| = \sum_{i=1}^{\frac{n}{2}} 4i - 2$. This can be rewritten as $\sum_{i=1}^{\frac{n}{2}} 4i - \sum_{i=1}^{\frac{n}{2}} 2$ and since $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, then $4\sum_{i=1}^{\frac{n}{2}} i = 4(\frac{1}{2}\frac{n}{2}(\frac{n}{2}+1))$ and $\sum_{i=1}^{\frac{n}{2}} 2 = n$ therefore $|\mathbb{B}^n| = n(\frac{n}{2}+1) - n = \frac{n^2}{2}$. □

For Magic Squares of odd order, the centre cell value, denoted M, is employed to form constraints on the row, column and diagonal constants. A similar approach is taken for Magic Squares of even order, but the value M to be used in the equivalent constraints is taken as the mean of the values in the paired cells of the centre subsquare of order 2.

Definition 4.14. *A SCMS of order n, n even, has value M equal to half the sum of the values in paired cells.*

11	193	179	37
79	137	151	53
157	59	73	131
173	31	17	199

Figure 4.6: *A PSCMS of Order 4 Where* $M = 105$ *[26]*

Unlike SCMS of odd order, not all pairs of complement values reside in the borders as two pairs are in the centre subsquare of order 2.

Lemma 4.15. *An associative Magic Square of order 4 has magic constant $S_4 = 4M$ and has eight pairs of complement values.*

Proof. From Definitions 1.5, 4.9 and 4.14, $a_{22}+a_{33} = a_{23}+a_{32} = a_{11}+a_{44} = a_{14}+a_{41} = 2M$, hence the main diagonals of the SCMS of order 4 sum to $4M$. For the order 4 grid to be a Magic Square, all rows, columns and main diagonals must equal the same constant, S_4, hence $S_4 = 4M$. □

Lemma 4.16. *A SCMS of order n, n even, with the value M of half the sum of the pairs of complement values (Definition 4.14), has magic constant* $S_n = nM$*.*

Proof. By Lemma 4.15, $S_4 = 4M$. Assume that for some $n = k$, $k > 4$ and even, $S_k = kM$. Now consider the case $n = k + 2$. As the values in each column sum to the magic constant, S_{k+2}, take the first and the $(k+2)^{th}$ columns, then $a_{11} + a_{(k+2)(k+2)} + a_{(k+2)1} + a_{1(k+2)} + \sum_{i=2}^{k+1} a_{i1} + \sum_{i=2}^{k+1} a_{i(k+1)} = 2S_{k+2}$. Consider the pairs $a_{11}, a_{(k+2)(k+2)}$ and $a_{(k+2)1}, a_{1(k+2)}$ which form diagonals with cells in the subsquare of order k; these all sum to S_{k+2}. Likewise the pairs $a_{i1}, a_{i(k+2)}, \forall i = 2, \ldots, k+1$, which form the centre rows with the subsquare of order k; these also sum to S_{k+2}. Since $S_k = kM$, then $a_{11} + a_{(k+2)(k+2)} = a_{(k+2)1} + a_{1(k+2)} = S_{k+2} - kM$ likewise $a_{i1} + a_{i(k+2)} = S_{k+2} - kM$, $i = 2, \ldots, k+1$.
Hence, $2S_{k+2} = (k+2)(S_{k+2} - kM)$ and therefore $S_{k+2} = (k+2)M$.
By induction the recurrence holds for any n even. Hence, for all n even, $S_n = nM$. □

Lemma 4.17. *Each border pair (Definition 4.11) of a SCMS of order n, n even, sums to* $2M$*.*

Proof. Proof follows immediately from Definition 4.14. □

Lemma 4.18. *A Strictly Concentric Magic Square of order n,* $n \geq 6$ *and even, cannot be associative.*

Proof. The positions of paired cells in a Strictly Concentric Magic Square of even order are provided in Definition 4.9, and from Lemma 4.17 the values in these paired cells have the same sum. Assume for contradiction that the Magic Square is associative; the pairs symmetric about the centre sum to the same value. To fulfil both of these requirements there need to be repeated values in cells. Therefore, a Strictly Concentric Magic Square of order n, $n > 4$ and even, cannot be associative. □

Every border of order $n \geq 6$, n even, of a SCMS of order n, n even, can undergo the permutations given for SCMS of odd order in Table 1.1 to form equivalent SCMS. Figure 4.7

shows two PSCMS which are equivalent, in which Figure 4.7(a) undergoes a permutation of the border pairs in rows 2 and 4 to form Figure 4.7(b).

13	109	113	149	163	83
29	11	193	179	37	181
103	79	137	151	53	107
167	157	59	73	131	43
191	173	31	17	199	19
127	101	97	61	47	197

(a) A PSCMS

13	109	113	149	163	83
167	11	193	179	37	43
103	79	137	151	53	107
29	157	59	73	131	181
191	173	31	17	199	19
127	101	97	61	47	197

(b) A PSCMS Equivalent to (a)

Figure 4.7: *Two Equivalent PSCMS of Order 6 [26]*

4.2.2 Minimum PSCMS of Order 6

As stated in Section 4.2.1, only associative subsquares of order 4 are considered in this thesis. This section explores the minimum PSCMS of order 6 with an associative subsquare of order 4.

4.2.2.1 Introduction

A SCMS of order 6 comprises eighteen distinct pairs of primes summing to 2M. Two of these pairs form the centre subsquare of order 2, six form the border $\mathbb{B}_4$ of the magic subsquare of order 4 and ten form the outer border $\mathbb{B}_6$. One such border example is shown in Figure 4.8(a), with one border pair shaded grey. By Lemma 4.16, a SCMS of order 6 has magic constant $S_6 = 6M$.

Recalling Section 1.1, a Magic Square of order 2 does not exist, hence the subsquare of order 2 in this analysis is not a magic subsquare. Recall from Definition 1.17, a minimum PSCMS has minimum M value. A minimum PSCMS of order 6 is given in [25] to have $S_6 = 630$ with

$M = 105$, and border pairs summing to $2M = 210$. The magic constant for the subsquare of order 4 is $4M = 420$, as is evident in Figure 4.8(b). It is proved by the current author in Lemma 4.19 that the minimum PSCMS of order 6 has $M = 105$ and hence from Lemma 4.16, $S_6 = 630$ and therefore the placement of primes in Figure 4.8 form a minimum PSCMS of order 6.

13	109	113	149	163	83
29					181
103					107
167					43
191					19
127	101	97	61	47	197

(a) Border of a Minimum PSCMS of Order 6

	11	193	179	37	
	79	137	151	53	
	157	59	73	131	
	173	31	17	199	

(b) Magic Subsquare of Order 4 for a Minimum PSCMS of Order 6

Figure 4.8: *Border and Magic Subsquare of Order 4 for a Minimum PSCMS of Order 6 [26]*

Recall that $\mathbb{P}$ denotes the set of all prime numbers and $\mathbb{P}'$ any subset of $\mathbb{P}$.

Lemma 4.19. *The minimum PSCMS of order 6 has $M = 105$, and hence $S_6 = 630$.*

Proof. Assume for contradiction that the M value is less than 105. Let $\mathbb{P}$ be the set of all prime numbers. To construct a PSCMS for which $M < 105$ there must exist at least eighteen distinct pairs of complement primes $x_i, \bar{x}_i \in \mathbb{P}'$ such that $x_i + \bar{x}_i = 2M$, where $x_i, \bar{x}_i$ are the values in paired cells. Two of these pairs form the subsquare of order 2, six of these pairs form the border of the subsquare of order 4, and ten form the outer border pairs of the PSCMS of order 6. There is no smaller M value for which there exist eighteen or more such pairs summing to $2M$. It is known that a PSCMS of order 6 with M value 105 exists. Hence, any PSCMS of order 6 with $M = 105$ and (from Lemma 4.16) $S_6 = 630$ is a minimum PSCMS, and one example is given in Figure 4.9. □

13	109	113	149	163	83
29	11	193	179	37	181
103	79	137	151	53	107
167	157	59	73	131	43
191	173	31	17	199	19
127	101	97	61	47	197

Figure 4.9: *A Minimum PSCMS of Order 6 [26]*

4.2.2.2 Construction of Minimum PSCMS of Order 6

Having established M for all minimum PSCMS of order 6, this section details the construction of such a square. Given the value $M = 105$, there are nineteen pairs of complement primes satisfying Lemma 4.17.

All minimum PSCMS of order 6 consist of eighteen pairs of complement primes formed from the following set of 38 prime numbers: $\mathbb{P}' = \{11, 13, 17, 19, 29, 31, 37, 43, 47, 53, 59, 61, 71, 73, 79, 83, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199\}$. Therefore, all squares require all apart from one pair from the list.

An Associative Prime Magic Square of order 4 for any given M, if one exists, can be generated using Algorithm 5. This uses a slightly different process to Algorithms 2 and 3, used for the generation of grids of odd order, as the border of order 4 is built around a subsquare of order 2 which is not a magic subsquare, requiring additional constraints to ensure that all rows and columns satisfy the magic constant.

Algorithm 5 *Algorithm to form an associative PMS of order 4*

Begin
Input M and form $\mathbb{P}'$, the set of all primes that form pairs summing to $2M$.
repeat
  Take two complement pairs of primes from $\mathbb{P}'$ and place into the order 2 subsquare of an empty grid of order 4 in paired cells.
  Form a set Q of the primes from $\mathbb{P}'$ not used in the subsquare.
  repeat
    Take a set of four distinct non-paired primes from Q that sum to $4M$, to form a set S, and their complements to form a set $\bar{S}$.
    Take a set of two distinct non-paired primes from Q to form a set T, and their complements to form a set $\bar{T}$.
    repeat
      Take an element x of S, and an element y of $\bar{S}$ that is not paired with x.
      Take elements s_2 and s_3 from S such that $s_2 \neq s_3 \neq x$ and neither are paired with y.
      Take elements t_1 and t_2 from T.
      Let the values in the subsquare of order 2 be $a_{22}, a_{23}, a_{32}, a_{33}$.
      Let R_1 be the sum of x, t_1, t_2 and y.
      Let R_2 be the sum of s_2, a_{22}, a_{23}, $\bar{s}_3$.
      Let R_3 be the sum of s_3, a_{32}, a_{33}, $\bar{s}_2$.
      Let C_2 be the sum of t_1, a_{22}, a_{32}, $\bar{t}_2$.
      Let C_3 be the sum of t_2, a_{23}, a_{33}, $\bar{t}_3$.
    until $R_1 = R_2 = R_3 = C_2 = C_3 = 4M$, or no further combinations of x, y, t_i, s_i are possible.
  until $R_1 = R_2 = R_3 = C_2 = C_3 = 4M$, or no further combinations of S are possible.
until $R_1 = R_2 = R_3 = C_2 = C_3 = 4M$, or no further order 2 subsquares are possible.
if $R_1 = R_2 = R_3 = C_2 = C_3 = 4M$ **then**
  Begin
  Place x in $(1,1)$, y in $(1,4)$ removing them from S and $\bar{S}$, and place their complements in the paired cells, removing them from $\bar{S}$ and S.
  Place the elements of T in $(1,2)$, $(1,3)$ in any order, and place their complements from $\bar{T}$ in the paired cells.
  Place the remaining elements of S in $(2,1)$, $(3,1)$ in any order, and place their complements from $\bar{S}$ in the paired cells.
  End
else
  No PSCMS exists for the input value of M.
end if
End

Algorithm 6 uses Algorithm 5 to form a subsquare of order 4, with chosen M and $\mathbb{P}'$, before using a similar process to Algorithms 2 and 3 to construct a PSCMS of order 6 around the given subsquare, if one exists.

Algorithm 6 *Algorithm to form PSCMS of order 6*

Begin
Input M and form $\mathbb{P}'$, the set of all primes that form pairs summing to $2M$.
repeat
    Construct a Prime Magic Square of order 4 with chosen M and $\mathbb{P}'$ using Algorithm 5.
    Place the magic subsquare of order 4, with chosen M value, into an empty grid of order 6.
    Form a set Q of the primes from $\mathbb{P}'$ not used in the subsquare.
    repeat
        Take a set of six distinct non-paired primes from Q that sum to $6M$, to form a set S, and their complements to form a set $\bar{S}$.
        Take a set of four distinct non-paired primes from Q to form a set T, and their complements to form a set $\bar{T}$.
        repeat
            Take an element x of S, and an element y of $\bar{S}$ that is not paired with x.
            Let X be the sum of x, y and the elements of T.
        until $X = 6M$, or no further combinations of x, y are possible.
    until $X = 6M$, or no further combinations of S are possible.
until $X = 6M$, or no further Prime Magic Squares of order 4 can be generated using Algorithm 5.
if $X = 6M$ **then**
    Begin
    Place x in $(1,1)$, y in $(1,6)$ removing them from S and $\bar{S}$, and place their complements in the paired cells, removing them from $\bar{S}$ and S.
    Place the elements of T in $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$ in any order, and place their complements from $\bar{T}$ in the paired cells.
    Place the remaining elements of S in $(2,1)$, $(3,1)$, $(4,1)$, $(5,1)$ in any order, and place their complements from $\bar{S}$ in the paired cells.
    End
else
    No PSCMS exists for the placed centre subsquare.
end if
End

Theorem 4.20. *A minimum PSCMS of order 6 with $M = 105$ is always formed using Algorithm 6 when a valid prime magic subsquare of order 4 is placed in the centre of the grid.*

Proof. For $M = 105$, the set of complement primes (primes summing to 210) $\mathbb{P}' = \{11, 13, 17, 19, 29, 31, 37, 43, 47, 53, 59, 61, 71, 73, 79, 83, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199\}$. Recall from Definition 2.3 that all border pairs are pairs of complement primes. Sixteen of these primes must be used in the subsquare and removed from $\mathbb{P}'$ to form Q.

In each case twenty of the remaining twenty-two primes are used in the border of order 6. Using a valid subsquare of order 4 there are six distinct pairs of complement primes in Q which satisfy conditions (1) and (3) below. From the remaining primes in Q it is possible to check that there are always 4 more pairs of complement primes that can be chosen to satisfy conditions (2) and (4) below.

(1) $a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} = 630$

(2) $a_{11} + a_{21} + a_{31} + a_{41} + a_{51} + a_{61} = 630$

(3) $a_{61} + a_{62} + a_{63} + a_{64} + a_{65} + a_{66} = 630$

(4) $a_{16} + a_{26} + a_{36} + a_{46} + a_{56} + a_{66} = 630.$

The primes are placed in the manner specified in Algorithm 6 (with the subsquare of order 4 having been determined in line 4, through a call to Algorithm 5, the paired values being determined in lines 7 to 15, and the border of the PSCMS of order 6 then being filled by lines 18 to 20). A minimum PSCMS is thereby formed. □

In Sections 2.2.3 and 4.1.1.3 an enumeration is given for the minimum PSCMS of order 5 and order 7 respectively. However, enumeration is not provided here for the minimum PSCMS of order 6. For PSCMS of even order, the order 2 subsquare centre can be formed in many different ways, in contrast to the single cell of PSCMS of odd order. This leads to a far larger number of cases to be considered in any enumeration, and this work is outside the scope of the thesis. Such enumeration is the focus of further work.

4.2.2.3 Conclusion of Minimum PSCMS of Order 6

The minimum PSCMS of order 6 with associative magic subsquare of order 4 is given with magic constant $S_6 = 630$, along with the possible pairs that can be used in the grid. An algorithm is given for construction as long as the magic subsquare of order 4 is associative. The enumeration even at this small order is difficult without substantial computing power, due in part to the fact that the subsquare of order 4 can undergo many more permutations than the squares of odd order. The subsquare of order 4 forms the basis for the addition of borders for PSCMS of order n, n even, when $n \geq 6$. Makarova [25] postulates that there exists a minimum PSCMS of order 6 with a non-associative magic subsquare of order 4 with a smaller M value but one is not given.

4.2.3 Construction of PSCMS of Even Order

Algorithm 7 uses Algorithm 5 to form a subsquare of order 4, with chosen M and $\mathbb{P}'$, before using a similar process to Algorithm 6 to construct a PSCMS of order n, n even, around the given subsquare, if one exists.

Algorithm 7 *Algorithm to form a PSCMS of order n, n even*

Begin
Input M and form $\mathbb{P}'$, the set of all primes that form pairs summing to $2M$.
repeat
  Construct a PSCMS of order 4 with chosen M and $\mathbb{P}'$ using Algorithm 5.
  Place the magic subsquare of order 4, with chosen M value, into the centre of an empty grid of order n.
  Form a set Q of the primes from $\mathbb{P}'$ not used in the subsquare.
  Let $i = 4$.
  repeat
    repeat
      Let $i = i + 2$.
      Take a set of i distinct non-paired primes from Q that sum to iM, to form a set S, and their complements to form a set $\bar{S}$.
      Take a set of $i - 2$ distinct non-paired primes from Q to form a set T, and their complements to form a set $\bar{T}$.
      repeat
        Take an element x of S, and an element y of $\bar{S}$ that is not paired with x.
        Let X be the sum of x, y and the elements of T.
      until $X = iM$, or no further combinations of x, y are possible.
    until $X = iM$, or no further combinations of S are possible.
  until $X = iM$, or no further PSCMS of order 4 can be generated.
  if $X = iM$ **then**
    Begin
    Place x in $(1, 1)$, y in $(1, i)$ removing them from S and $\bar{S}$, and place their complements in the paired cells, removing them from $\bar{S}$ and S.
    Place the elements of T in $(1, 2)$,..., $(1, i - 1)$ in any order, and place their complements from $\bar{T}$ in the paired cells.
    Place the remaining elements of S in $(2, 1)$,..., $(i - 1, 1)$ in any order, and place their complements from $\bar{S}$ in the paired cells.
    End
  else
    No PSCMS exists for the placed centre subsquare.
  end if
until $i = n$.
End

For a PSCMS of order n, $n \geq 6$ the number of magic subsquares of order 4 depends on M and therefore in general, it is not known how many are valid for a given PSCMS.

Theorem 4.21. *A PSCMS of order n, n even, with given M value is always formed using Algorithm 7 when every subsquare of order $n - 2i$, $i = 1, \ldots, \dfrac{(n-4)}{2}$, is a valid PSCMS.*

Proof. For a given M, the complement primes (primes summing to $2M$) form a set $\mathbb{P}'$. The set must have cardinality $|\mathbb{P}'| \geq n^2$. Now, $n^2 - 4n - 4$ of these primes must be used in the subsquare and removed from $\mathbb{P}'$ to form a set Q. Then $4(n-1)$ primes from Q are used in the border of order n. Given that the subsquare of order $n-2$ is a valid PSCMS, there are n distinct pairs of complement primes in Q which satisfy conditions (1) and (3) below. From the remaining primes in Q it is possible to check that there are always $n-2$ more pairs of complement primes that can be chosen to satisfy conditions (2) and (4) below.

(1) $a_{11} + a_{12} + \cdots + a_{1n} = nM$

(2) $a_{11} + a_{21} + \cdots + a_{n1} = nM$

(3) $a_{n1} + a_{n2} + \cdots + a_{nn} = nM$

(4) $a_{1n} + a_{2n} + \cdots + a_{nn} = nM$

The primes are placed in the manner specified in Algorithm 7 (with the subsquare of order 4 having been determined in line 4, through a call to Algorithm 5, the paired values being determined in lines 9 to 18, and the border of the PSCMS of order n then being filled by lines 21 to 23). A PSCMS is thereby formed. □

4.3 Conclusion of PSCMS of Higher Order

In this chapter PSCMS of higher order are formally explored for the first time. Initially, properties of PSCMS of odd order are generalised, building on the definitions from Chapters 1 and 2. The minimum PSCMS of order 7 is then considered using the framework for the minimum PSCMS of order 5 and an algorithm for construction presented. The centre cell value of the minimum PSCMS of order 7 is 641. It is here established that there are six possible non-equivalent grids of order 3 that are valid subsquares which facilitates the enumeration of 3,043,905,984,921,600 minimum PSCMS of order 7, 11,468,227 of which are non-equivalent. Finally in this section, an algorithm for the construction of PSCMS of general odd order is given.

This is followed by an introduction to SCMS of even order, with definitions which differ from the odd order definitions provided here, to establish properties for all SCMS of even order with an associative magic subsquare of order 4. Although some of the concepts are similar, structurally odd and even squares are very different. An introduction to the minimum PSCMS of order 6, which has $M = 105$, is then given with two novel algorithms for construction provided. The first constructs an associative magic subsquare of order 4 and the second constructs a PSCMS of order 6 around the given subsquare. An algorithm is then presented for constructing PSCMS of general even order.

Chapter 5

Water Retention

5.1 Literature Review

The water retention problem in relation to Number Squares conceptualises the values assigned to cells as blocks of height proportionate to the values. Water is then ‘poured’ over the grid and is only retained in those cells which are surrounded, horizontally and vertically, by cells which are taller [18].

The idea of water retention on a Magic Square was postulated by Knecht in 2007 [18] and in 2010 Zimmermann held an online programming contest to try and find the maximum water retention a Normal Magic Square (NMS) could contain [45]. Proposed solutions were found on Magic Squares up to order 28 by various mathematicians around the world, with Trump being the overall winner. This contest and the results found were the benchmark for a paper by Öfverstedt [12] in 2012 which discussed maximizing water retention as the order of the Magic Squares increased. Achieving maximum water retention becomes increasingly difficult as the number of existing grids increases. Öfverstedt uses a constraint-based local search in order to tackle the problem in a non-exhaustive way.

In the same year (2012), Knecht, Trump, ben-Avraham and Ziff [21] reported on the retention capacity of random surfaces. This was based on water retention on a Magic Square, but the focus instead was on bounded surfaces with varying heights. They looked at differently levelled systems and explained the behaviour using percolation theory for which the reader is referred to [21] for further details. The authors presented the average amount of water retained on grids up to order 60, for n-level system where $n = 2, \ldots, 8$. For example, when $n = 2$, a 2-level system has a uniform distribution of heights 0 and 1 across the surface whereas a 3-level system would include heights of 0, 1 and 2. As the order of the grid increases the average amount of water retained increases exponentially for all n-level systems. However, it is noted that when the order is greater than 51 the 2-levelled system retains, on average, more water than the 3-levelled system.

Baek and Kim [4], also in 2012, used percolation theory to find the critical condition of the water retention model. This led to an upper bound of water retention in a random landscape which the authors note, if "the global landscape on the Earth can be roughly regarded as random, we may retrospectively understand that too much of the Earth is covered by water to be believed to be a flat disk with steep cliffs at the boundary as once believed."

In 2014 Schrenk, Araujo, Ziff and Herrmann [32] published an extension of Knecht et al's water retention model to correlated random surfaces. They extended the research to study the impact of correlating the heights in the landscape and showed that long range correlations decrease the retention capacity. White [41] has written a multitude of programming tools in order to formulate different kinds of Magic Squares, and one of these includes a water retention program which allows the user to upload a Magic Square and check where water is retained. The program then enables the permutation of cells in order to compare the water retention.

In 2019 Hasan and Polash [13] published a paper for maximizing water retention on Magic Squares which uses the same approach as Öfverstedt, constraint-based local search. They note that their program gives them close to the maximum known retention and is more efficient than previous algorithms.

This chapter gives an introduction to the concept of water retention on Number Squares with relevant definitions and basic examples given. This concept is applied to Normal Number Squares and then NMS with a comparison of maximum retention given. The idea of a maximal lake is introduced and implemented on Normal Number Squares, with a comparison given of the maximum water retention on the grid and the water retention of a maximal lake. Finally maximal lakes on Normal Prime Squares is given followed by the maximum water retention on the minimum PSCMS of order 5 presented in Chapter 2.

5.2 Introduction

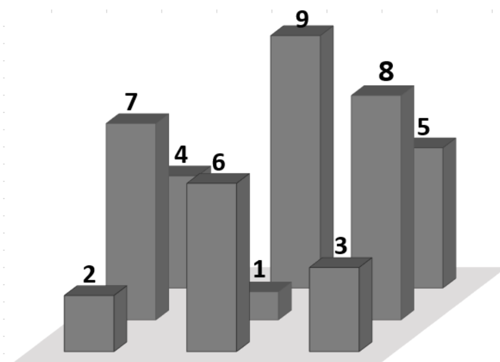
Water retention is here considered on Normal Number Squares, Normal Magic Squares, Prime Number Squares and Prime Strictly Concentric Magic Squares.

Definition 5.1. *A **Number Square** of order n is an n by n grid containing n^2 distinct integers. A **Normal Number Square** contains the integers 1 to n^2. A **Prime Square** of order n is an n by n grid containing n^2 distinct primes. A **Normal Prime Square** contains the first n^2 primes.*

Knecht collated the maximum water retention on Normal Number Squares of order n, $n \leq 30$, in 2015 [19] and it is his notation which is used throughout this chapter. Figure 5.1 shows a Normal Number Square of order 3 with maximum water retention and a visual representation of these cells as block heights; the equation to quantify the amount of water retained is given in Section 5.2.1.

2	6	3
7	1	8
4	9	5

(a) Order 3 Normal Number Square



(b) Representation of Heights as Blocks

Figure 5.1: *A Normal Number Square of Order 3 with Maximum Water Retention*

5.2.1 Islands, Lakes and Ponds

Water can be retained on grids in many different patterns and these patterns are referred to as islands, lakes and ponds using definitions given by Knecht and Trump [20].

Definition 5.2. *A **lake** is a body of water on a grid of order n which reaches the dimensions $(n-2)$ by $(n-2)$ where all cells in the body of water are horizontally or vertically contiguous.*

Figure 5.2(a) shows the largest possible lake pattern on a grid of order 4, and the smallest in Figure 5.2(b).

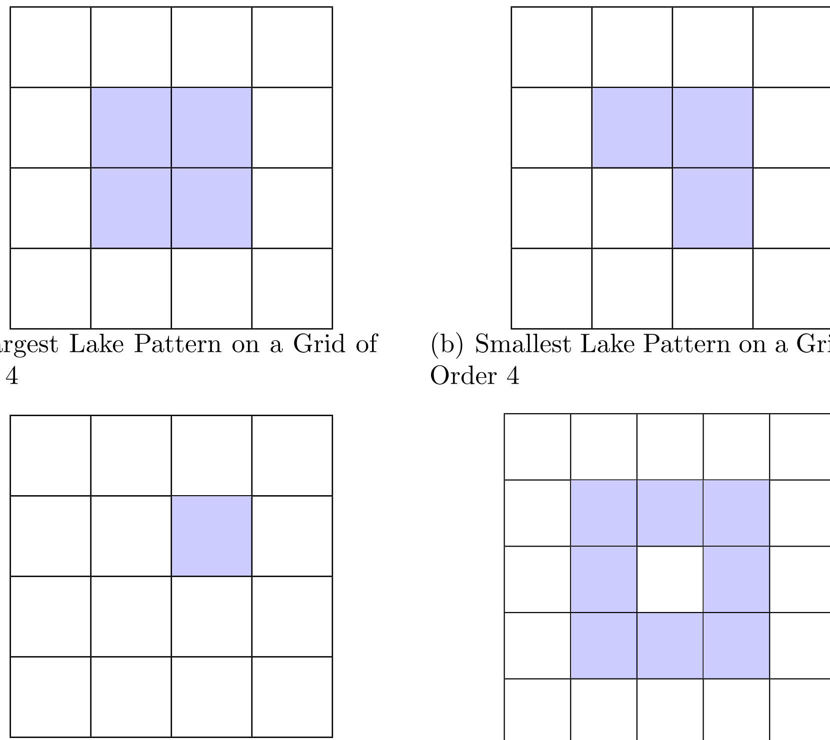
Definition 5.3. *A **pond** is any body of water which does not fulfil the size requirements of a lake.*

There can be just one pond on a grid or there can be multiple and the larger the grid the more possibilities there are for multiple ponds. Figure 5.2(c) shows a pond pattern of one cell on a grid of order 4.

Definition 5.4. *A **spillway** is defined to be the smallest integer directly horizontal or vertical to a cell, or group of cells, which retains water.*

Definition 5.5. *An **island** describes a cell, or a collection of cells that are horizontally or vertically contiguous, which do not contain water, within an area of water. An island occurs when a cell, or collection of cells, inside a lake or a pond have values higher than the spillway.*

Figure 5.2(d) shows an island pattern of one cell inside a lake pattern on a grid of order 5.



(a) Largest Lake Pattern on a Grid of Order 4

(b) Smallest Lake Pattern on a Grid of Order 4

(c) A Pond Pattern on a Grid of Order 4

(d) An Island of Order 1 Pattern Inside a Lake Pattern

Figure 5.2: *Water Retention Patterns*

Let s denote the value of a spillway, w denote the number of water retaining cells this spillway applies to and the base b denote the sum of the values inside the water retaining cells. Then the volume of water retained V is given in Equation 5.1 and can be used for each body of water on any Number Square [22].

$$V = (s \times w) - b \tag{5.1}$$

Where the grid has more than one spillway, the volume of water retained is the sum of the volumes in each body of water.

The maximum water retention on a grid is the maximum amount of water it can hold, and this value has been determined on Normal Number Squares up to $n = 30$. This information has been collated by Knecht [19] and is available on the OEIS, Knecht comments that a program has been written by Trump to calculate the maximum water retention of a Normal Number Square up to order 250. Maximum water retention on NMS can be found using similar techniques, and an example for each case up to $n = 28$ has been collated by Zimmermann, based on the contest he held, and is available at [45]. There were improvements made to the volume of water retention on some grids after the competition ended and these are also detailed, and given that there have been no further documented improvements since 2010 these results for maximum water retention on Magic Squares are, in this thesis, considered final. The results up to order 9 are given on the OEIS [29]. Table 5.1 shows the maximum water retention for both Normal Number and Normal Magic Squares for order n where $n = 3, \ldots, 15$.

Order (n)	**Normal Number Square**	**Normal Magic Square**
3	5	0
4	26	15
5	84	69
6	222	192
7	488	418
8	946	797
9	1664	1408
10	2723	2267
11	4227	3492
12	6277	5185
13	8993	7445
14	12514	10397
15	16976	14154

Table 5.1: *Comparison of Maximum Water Retention Between Normal Number Squares and Normal Magic Squares [19] [29] [45]*

Figure 5.3(a) shows the maximum water retention on a NMS of order 5 which occurs by forming a lake, shown in blue and the spillway in yellow. The maximum water retention on a NMS of order 6 is shown in Figure 5.3(b) which occurs by forming two ponds, shown in blue with the spillways in yellow.

3	12	22	17	11
16	18	7	4	20
25	2	14	5	19
8	23	1	24	9
13	10	21	15	6

(a) A Normal Magic Square of Order 5 with Maximum Water Retention

8	15	22	26	24	16
20	33	30	2	3	23
32	4	13	36	1	25
29	5	11	14	31	21
12	35	7	6	34	17
10	19	28	27	18	9

(b) A Normal Magic Square of Order 6 with Maximum Water Retention

Figure 5.3: *Two Normal Magic Squares with Maximum Water Retention [45]*

Definition 5.6. *Consider a grid of order* n*, then if every cell in the subsquare of order* $(n-2)$ *retains water, this is referred to as a* ***maximal lake****.*

Through the placement of values, a Normal Number Square of any order can be formed to contain a maximal lake. Table 5.2 shows a comparison between the water retention when a Normal Number Square is constructed to form a maximal lake, which is further explored in Section 5.3, and the maximum water retention possible on a Normal Number Square by forming lakes and ponds.

Order	Maximal Lake	Maximum Retention
3	5	5
4	26	26
5	81	84
6	200	222
7	425	488
8	810	946
9	1421	1664
10	2336	2723
11	3645	4227
12	5450	6277
13	7865	8993
14	11016	12514
15	15041	16976

Table 5.2: *Comparison of Maximal Lake Retention and Maximum Possible Retention on Normal Number Squares [19]*

5.3 Normal Number Squares

Consider a Normal Number Square of order n which contains a maximal lake, i.e. all cells in the subsquare of order $(n-2)$ contain water. This can be formed, with maximum water retention using Construction 5.7.

Construction 5.7. *Given an empty grid of order n, place the $(n-2)^2$ smallest values in the centre subsquare of order $(n-2)$. Place the next four smallest values in the corners of the grid. The remaining values may be placed in the empty edge cells arbitrarily.*

This construction ensures the maximum possible difference between the values in the water retaining cells and the spillway, in order to have maximum possible water retention given the structure. The value of the spillway for a Normal Number Square of order n is $s = (n-2)^2+5$.

Example 5.8. *Figure 5.4 shows a maximal lake formed using Construction 5.7 on a grid of order 6. The water retaining cells are shown in blue and the spillway is shown in yellow.*

17	21	22	23	24	18
25	1	2	3	4	26
27	5	6	7	8	28
29	9	10	11	12	30
31	13	14	15	16	32
19	33	34	35	36	20

Figure 5.4: *A Normal Number Square of Order 6 with a Maximal Lake*

Theorem 5.9. *Given a Normal Number Square of order n, a maximal lake with maximum water retention has* $V = \frac{1}{2}(n-2)^2(n^2-4n+13)$.

Proof. From Equation 5.1, $V = sw - b$, using Construction 5.7, $s = (n-2)^2+5$, $w = (n-2)^2$ and b is the sum of the first $(n-2)^2$ integers. Since $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$, $b = \sum_{r=1}^{(n-2)^2} r = \frac{1}{2}(n-2)^2((n-2)^2+1)$. Hence, $V = ((n-2)^2+5)((n-2)^2) - (\frac{1}{2}(n-2)^2((n-2)^2+1)) = \frac{1}{2}(n-2)^2(n^2-4n+13)$. □

Figure 5.4 shows a maximal lake formed using Construction 5.7 on a Normal Number Square of order 6 where $V = 200$.

When n is greater than 4, the maximum water retention is not obtained by forming a maximal lake. Shown in Table 5.2, as n increases, the maximum water retention on a Normal Number Square is approximately 1.1 times that of the maximum water retention of a maximal lake.

Example 5.10. *Figure 5.5 shows two Normal Number Squares of order 5, the first of which is a grid with a maximal lake 5.5(a) where* $V = 81$*, and the second of which is a grid with maximum water retention 5.5(b) where* $V = 84$*. The cells containing water are given in blue and the spillways are yellow.*

10	14	15	16	11
25	1	2	3	17
24	4	5	6	18
23	7	8	9	19
13	22	21	20	12

(a) A Normal Number Square of Order 5 with a Maximal Lake

8	9	23	22	11
10	24	1	2	21
25	3	4	5	20
13	16	6	7	19
14	15	17	18	12

(b) A Normal Number Square of Order 5 with Maximum Water Retention [19]

Figure 5.5: *Comparison of Water Retention on Normal Number Squares of Order 5*

5.4 Normal Magic Squares

Given the added constraints of a Magic square, the maximum water retention (shown in Table 5.1) is approximately 0.8 times that of the maximum water retention of a Normal Number Square. There exists only one NMS of order 3 up to equivalence, given in Figure 5.6(a). Since $M = 5$ and the pairs of cells adjacent to the centre sum to $2M$, each row and column contains an integer smaller than five and therefore a NMS of order 3 cannot retain water. The maximum water retention of a Normal Number Square of order 4, using Theorem 5.9, is 26. However, the maximum water retention of a NMS of order 4 is 15, with the difference being due to the strict positioning of the values in order to satisfy the constraints of the magic constant [45]. The maximum water retention on a NMS of order 4 is not obtained by forming a maximal lake, or any smaller lake, instead there are two ponds on the grid shown in Figure 5.6(b). The list of comparisons up to order 15 can be seen in Table 5.1.

Example 5.11. *The single Normal Magic Square of order 3 is given in Figure 5.6(a). The maximum water retention on a Normal Magic Square of order 4 is shown in Figure 5.6(b), there are two ponds and the total water retained is 15 units.*

4	9	2
3	5	7
8	1	6

(a) The Normal Magic Square of Order 3

1	7	10	16
8	14	3	9
12	2	15	5
13	11	6	4

(b) A Normal Magic Square of Order 4 with Maximum Water Retention [45]

Figure 5.6: *The Normal Magic Square of Order 3 and A Normal Magic Square of Order 4 with Maximum Water Retention*

5.5 Prime Number Squares

Recall that, a Prime Number Square consists of all the properties of a Number Square but with the added constraint that all the entries must be prime numbers. To achieve the maximal lake where the prime numbers are sequential starting at 2 (Normal Prime Number Squares) Construction 5.7 can be used. This forms a maximal lake on the grid, where for a Prime Number Square of order n the spillway is the prime number in the sequential list in position $((n-2)^2+5)$. For example, in a Prime Number Square of order 4 the spillway is the 9^{th} prime number in the sequential list which is 23. Equation 5.1 can be used to calculate the water retention on a Prime Number Square and is applied to this case in Theorem 5.12.

Theorem 5.12. *Given a Normal Prime Number Square of order n, a maximal lake with maximum water retention has $V = x_{(n-2)^2+5} \times (n-2)^2 - \sum_{i=1}^{(n-2)^2} x_i$ where x_i refers to the prime number in the ith position in the ordered list of primes 1 to n^2.*

Proof. Follows directly from Theorem 5.9. □

Example 5.13. *Figure 5.7 shows a maximal lake on a Normal Prime Number Square of order 4, retaining 75 units of water.*

11	23	29	13
53	2	3	31
47	5	7	37
19	43	41	17

Figure 5.7: *A Prime Number Square of Order 4 with a Maximal Lake*

The volume of water retained, much like Normal Number Squares, is greatly reduced when the grid is magic. The maximum volume of water retained is unknown for Prime Number Magic Squares in general as there are no constraints on the primes that can be used. However, by giving additional structure to these grids it is possible to generalise the water retention in certain cases. In Sections 5.5.1 and 5.5.2 the maximum water retention on the minimum PSCMS of order 5 is explored. Equation 5.1 and the permutations from Table 1.1 are used to quantify the maximum amount of water retention on each of the types of minimum PSCMS of order 5 from Section 2.2. In Figure 5.8 an example of each type 1 minimum PSCMS of order 5 is given and in Figure 5.9 an example of each type 2 minimum PSCMS of order 5 is given; these PSCMS were explored in Chapter 2, Section 2.2.

5.5.1 Minimum PSCMS of Order 5 with Magic Subsquare 1

For the minimum PSCMS of order 5 with magic subsquare 1, there are six non-equivalent types which each use a different list of primes. Within these types there are variants which, although having the same list of primes, are non-equivalent and may have different water retention. Consider only the variant which retains maximum water retention, where several variants retain the same maximum volume of water, just one example is given. Results were obtained exhaustively by hand. Figure 5.8 shows each type with the cells retaining water in blue and the spillways in yellow.

Lemma 5.14. *The maximum amount of water retained on the minimum PSCMS of order 5 with magic subsquare 1 is:*

(1) 804 units for type 1A;

(2) 1002 units for type 1B;

(3) 792 units for type 1C;

(4) 792 units for type 1D;

(5) 804 units for type 1E;

(6) 804 units for type 1F.

Proof. All minimum PSCMS of order 5 with magic subsquare 1 are of types 1A,...,1F (given in Section 2.2.2). Consider only the variant in each type which retains the most water. Since all variants have been considered for each type there are no non-equivalent grids which retain more water and using Table 1.1, no equivalent grids to those given in Figure 5.8 contain more water. □

401	71	419	53	311
389	461	23	269	113
263	59	251	443	239
11	233	479	41	491
191	431	83	449	101

(a) Type 1A PSCMS Using A1 and A3 with 804 Units of Water Retained

191	53	419	353	239
11	461	23	269	491
401	59	251	443	101
389	233	479	41	113
263	449	83	149	311

(b) Type 1B PSCMS Using B1 and B2 with 1002 Units of Water Retained

401	113	419	11	311
353	461	23	269	149
239	59	251	443	263
71	233	479	41	431
191	389	83	491	101

(c) Type 1C PSCMS Using C3 and C4 with 792 Units of Water Retained

71	83	449	389	263
191	461	23	269	311
401	59	251	443	101
353	233	479	41	149
239	419	53	113	431

(d) Type 1D PSCMS Using D1 and D3 with 792 Units of Water Retained

401	71	419	11	353
389	461	23	269	113
263	59	251	443	239
53	233	479	41	449
149	431	83	491	101

(e) Type 1E PSCMS Using E1 and E2 with 804 Units of Water Retained

401	149	263	11	431
419	461	23	269	83
311	59	251	443	191
53	233	479	41	449
71	353	239	491	101

(f) Type 1F PSCMS Using F5 and F6 with 804 Units of Water Retained

Figure 5.8: *Maximum Water Retention For Type 1 Minimum PSCMS of Order 5*

The minimum PSCMS of order 5 with magic subsquare 1 that contains the highest volume of water is of type 1B. This has six water retaining cells, five of which form a lake and a single cell forms a pond. The grid with maximum retention of type 1D has the same pattern of retention as type 1B however the spillway of the lake is 311 rather than 353 which means it holds 210 units less of water. The most common volume of water to be retained is 804 units; this is a pattern of three single cell ponds such that each spillway is another cell in the subsquare of order 3.

5.5.2 Minimum PSCMS of Order 5 with Magic Subsquare 2

For the minimum PSCMS of order 5 with subsquare 2, there are nine non-equivalent types which each use a different list of primes. Within these types there are variants which although they use the same list of primes, are non-equivalent. The same process is used here as for the minimum PSCMS of order 5 with magic subsquare 1 in order to find the maximum water retention on each type. Figure 5.9 shows each type with the cells retaining water in blue and the spillways in yellow.

Lemma 5.15. *The maximum amount of water retained on the minimum PSCMS of order 5 with magic subsquare 2 is:*

(1) 696 units for type 2A;

(2) 912 units for type 2B;

(3) 696 units for type 2C;

(4) 648 units for type 2D;

(5) 696 units for type 2E;

(6) 696 units for type 2F;

(7) 696 units for type 2G;

(8) 696 units for type 2H;

(9) 696 units for type 2I.

Proof. All minimum PSCMS of order 5 with magic subsquare 2 are of types 2A,...,2I (given in Section 2.2.2). Consider only the variant in each type which retains the most water. Since all variants have been considered for each type there are no non-equivalent grids which retain more water and using Table 1.1, no equivalent grids to those given in Figure 5.9 contain more water. □

353	101	311	11	479
449	431	83	239	53
389	59	251	443	113
41	263	419	71	461
23	401	191	491	149

(a) Type 2A PSCMS Using A3 and A4 with 696 Units of Water Retained

101	41	491	389	233
53	431	83	239	449
479	59	251	443	23
353	263	419	71	149
269	461	11	113	401

(b) Type 2B PSCMS Using B3 and B4 with 912 Units of Water Retained

401	53	479	11	311
233	431	83	239	269
389	59	251	443	113
41	263	419	71	461
191	449	23	491	101

(c) Type 2C PSCMS Using C1 and C2 with 696 Units of Water Retained

491	149	191	23	401
233	431	83	239	269
389	59	251	443	113
41	263	419	71	461
101	353	311	479	11

(d) Type 2D PSCMS Using D1 and D2 with 648 Units of Water Retained

191	449	491	23	101
269	431	83	239	233
353	59	251	443	149
41	263	419	71	461
401	53	11	479	311

(e) Type 2E PSCMS with 696 Units of Water Retained

191	479	461	11	113
269	431	83	239	233
353	59	251	443	149
53	263	419	71	449
389	23	41	491	311

(f) Type 2F PSCMS with 696 Units of Water Retained

353	101	479	11	311
269	431	83	239	233
389	59	251	443	113
53	263	419	71	449
191	401	23	491	149

(g) Type 2G PSCMS Using G3 and G2 with 696 Units of Water Retained

269	461	401	11	113
191	431	83	239	311
353	59	251	443	149
53	263	419	71	449
389	41	101	491	233

(h) Type 2H PSCMS Using H3 and H2 with 696 Units of Water Retained

449	113	269	23	401
311	431	83	239	191
353	59	251	443	149
41	263	419	71	461
101	389	233	479	53

(i) Type 2I PSCMS with 696 Units of Water Retained

Figure 5.9: *Maximum Water Retention For Type 2 Minimum PSCMS of Order 5*

The minimum PSCMS of order 5 with magic subsquare 2 that contains the highest volume of water is of type 2B. Similar to type 1B, this has six water retaining cells, five of which form a lake and a single cell forms a pond. There is no other type with subsquare 2 that has the same pattern of water retention. The most common volume of water to be retained is 696 units; this is a pattern of three single cell ponds such that each spillway is another cell in the subsquare of order 3.

5.5.3 Comparison of Maximum Water Retention on Minimum PSCMS of Order 5

The volume of water retained is substantially lower on the minimum PSCMS of order 5 with subsquare 2, the type with the highest volume of water, type 2B at 912 units, is 90 units lower than the grid with the highest volume of water for subsquare 1, type 1B with 1002 units.

In both cases the most common patterns of water retention are those of the three lowest cells in the subsquare retaining water and the three spillways also being in the subsquare; ten out of the fifteen types have this pattern. In this pattern, the grids with subsquare 1 retain 804 units of water, 108 units more than the grids with subsquare 2 which retain just 696 units. This is because the cells containing water in subsquare 1 have values 23, 41 and 59 which sum to 123, whereas the cells containing water in subsquare 2 have values 59, 71 and 83 summing to 213. Hence subsquare 1 has a lower base level and if the spillways were the same the grid would retain 90 units more of water. The spillways then on subsquare 1 have values 233, 251 and 443 which sum to 927 whereas the spillways on subsquare 2 have values 239, 251 and 419 which sum to 909. The spillways are higher on subsquare 1 allowing 18 units more of water. Hence, with this pattern, the grids with subsquare 1 retain 108 units more.

The actual patterns of water retention of the individual types rely on which values are in the non-corner edge cells, so a combination of the corner cells and the omitted pair from each list of primes affects the volume of water each grid can retain.

5.6 Conclusion of Water Retention

The idea of water retention on Number Squares is introduced in this chapter with some basic results for Normal Number Squares and NMS from the literature given. Notation is formally defined, and comparisons are given between maximum water retention on Normal Number

Squares and maximum water retention on NMS as well as maximum retention given specific patterns of cells retaining water. Prime Number Squares are then introduced with the focus being the minimum PSCMS of order 5 from Chapter 2. It is found that those PSCMS with magic subsquare 1 contain more water than those with magic subsquare 2 due to the values in subsquare 1 allowing for a lower base and higher spillways.

Chapter 6

Conclusion

Chapter 1 of this thesis gives an overview of known concepts of Magic Squares as well as the first formal definitions of structure for Strictly Concentric Magic Squares and Prime Strictly Concentric Magic Squares. Relevant definitions for Latin Squares are taken from the literature and provided here for context before being applied in subsequent chapters to Magic Squares. There is a discussion on enumeration in the literature on Magic Squares which provides background as to how difficult enumeration is in general as well as providing context for the research in this thesis.

A known minimum PSCMS of order 5 is given in Chapter 2 along with notation and additional structural properties for a SCMS. This chapter uses the concept of Strictly Concentric Magic Squares of order n, n odd, with magic constant $S_n = nM$, being built from a centre cell M through the addition of successive borders of each larger order grid, and provides the first formal analysis of minimum PSCMS of order 5. A proof of the minimum PSCMS of order 5 having magic constant $S_5 = 1255$ is provided with the first classification into types of all grids satisfying this minimum based on the subsquare of order 3 and the list of primes, $\mathbb{P}'$, associated with M. It is noted that there are two non-equivalent subsquares of order 3 and each has different possible borders which, along with allowable permutation operations, facilitates the first enumeration of the 80,640 grids of which 35 are non-equivalent.

Concepts already established for Latin Squares and Sudoku grids form the basis of Chapter 3 with completable partial SCMS of odd order being formally defined. The cardinality of the minimal forced completable set, the minimum number of triples specifying the non-empty cells of a grid in order for it to be strongly completable, is established for SCMS of odd order. A bound is established for the cardinality of a minimal critical set, the minimum number of non-empty cells for the grid to be uniquely completable, either strongly or weakly, for SCMS of odd order. The minimal critical set is then found on the minimum PSCMS of order 5 from Chapter 2. Unavoidable sets in SCMS are defined in this thesis and the different forms explored on individual minimum PSCMS of order 5 from Chapter 2. The results of this chapter are a significant contribution to the literature on Magic Squares, as no formal treatment of Strictly Concentric Magic Squares has previously been published. This work provides a framework and definitions useful for further work in this area, in addition to the specific results provided.

Chapter 4 uses the definitions and concepts from Chapters 1 and 2 and these are applied to PSCMS of order n, $n > 5$. Initially, the construction and enumeration of PSCMS of order 7 are detailed. This leads to a construction of PSCMS of odd order in general. The former begins with a similar approach to that used in the analysis of the minimum PSCMS of order 5 but with six non-equivalent subsquares of order 3 and 162,436 potential subsquares of order 5 to give the total number of PSCMS of order 7 as 3,043,905,984,921,600 grids with a magic constant of 4,487. Given sufficient computing power, the same approach could theoretically be taken for Prime Strictly Concentric Magic Squares of higher order n, n odd, as long as the centre cell value M is known and a valid PSCMS can be formed. SCMS of even order are then introduced and definitions where they differ from odd order are given. Similar techniques for analysis, construction and enumeration can be used, noting that as a Magic Square of order 2 does not exist the Magic Square of order 4 must be formed first. An algorithm for construction of a PSCMS of even order is presented, and the minimum PSCMS of order 6 considered. In general, enumeration of PSCMS of even order is much harder than that of odd order, since there are more possibilities for the construction of the subsquare of order 4. Hence enumeration is not provided. Note, however, that after the initial subsquare of order 4 is formed, the borders are built up using the same techniques as for odd order, and could in theory be analysed in the same way.

Chapter 5 describes the concept of water retention on Number Squares, Prime Number Squares, Magic Squares and PSCMS. A literature review is provided on previous work on water retention on Number Squares and Magic Squares. Definitions from the literature are formalised and patterns of cells in which water is retained are explored. Maximum water retention on Normal Number Squares and Normal Magic Squares is given before maximum water retention is found on each type of minimum PSCMS of order 5 as given in Chapter 2.

6.1 Future Work

This thesis provides a first formal treatment of the structure, enumeration and analysis of properties of SCMS. As such, the foundational work and results presented suggest many avenues of future study.

Using the concepts from Chapters 1 and 2, PSCMS of order 5 with different centre cell values could be explored. Only the minimum PSCMS were investigated and so the same analysis could be carried out on PSCMS of order 5 having larger centre cell value, and hence different lists of primes.

More research could be conducted into the minimal critical sets on SCMS in order to determine whether an exact equation for the size of the minimal critical set based on the order can be found, or whether it always depends on the values in the cells in each grid. Minimal forced completable sets and minimal critical sets, as well as patterns of unavoidable sets, could be investigated in general for SCMS of all orders. The forms of unavoidable sets identified on minimum PSCMS of order 5 will exist on subsquares of PSCMS of higher order. It is here conjectured that additional forms may exist on PSCMS of higher order. It may also be interesting to explore whether these forms exist on other categories of Magic Squares.

The minimum centre cell values of PSCMS of order n, $n \leq 19$ and odd, are known. The number of pairs of complement primes as well as the number of potential subsquares of order 3 could be used to determine the rate of increase as the order of the square increases. This rate of increase could be used to predict the number of minimum PSCMS of order n, for any given value of n, n odd. Given sufficient computing power, full enumeration could in theory be achieved for PSCMS of higher order and minimums determined for $n > 19$ in order to establish whether there is an exponential relationship between order and the number of PSCMS. It is conjectured that:

Conjecture 6.1. *For a minimum PSCMS of order n, the centre cell value M is of the form $6k - 1$.*

Conjecture 6.2. *A PSCMS of order n, n odd, with fixed M, can be formed if there exist at least $\dfrac{n^2 - 1}{2}$ pairs of complement primes summing to $2M$.*

The focus of this thesis is the investigation of PSCMS of odd order. Much investigation is possible of PSCMS of even order, beyond the initial results provided in Chapter 4. Full enumeration of minimum PSCMS of order 6 could be completed with sufficient computing power using Algorithm 6.

Finally, a fuller analysis could be completed regarding water retention on Magic Squares. This may highlight common patterns leading to maximum water retention. Also, it would be worth investigating whether direct analysis of the lists of primes for each type of PSCMS could offer prediction of water retention, without requiring exhaustive analysis of the PSCMS.

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