A Prediction Based Approach to Output Consensus of Heterogeneous Multi-Agent Systems With Delays

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Abstract: The output consensus of multi-agent systems is investigated, where constant communication delay is present and the dynamics of the agents are heterogeneous. Based on the networked predictive control scheme, the distributed consensus protocol with dynamic output feedback controller is designed, and the sufficient conditions of the output consensus are obtained. Numerical examples illustrate the effectiveness of the proposed method.

1 Introduction

The fast development of embedded computation and wireless communication in recent decades have enabled and popularized a new control structure where multiple relatively independent devices could work together for some single objective, thus the name "multi-agent systems" (MASs). Practical examples of such a control structure can be seen in multi-robot systems, multi-satellite systems, air vehicle formation, underwater vehicle queue, and so on. For this new control structure some theoretical challenges are yet to be solved.

Recently, one often discussed topic for MASs is "consensus", the theoretical framework of which was first proposed by Olfati-Saber and Murray [1]. Following the line considerable works have been reported, mainly for homogenous MASs where each agent has the same dynamics. For instance, stabilizability and consensus of homogeneous MASs described by a positive state-space model are investigated in [2]. A robust control approach for an observer-type protocol is presented by virtue of low-gain and high-gain techniques in [3]. However, many practical systems should be modelled in the heterogeneous fashion in reality, since there is often no way to assume the same dynamics for each agent. But unlike its homogeneous counterpart, heterogeneous MASs still face many challenges.

As is known to all, communication delay is something that the system can not avoid [4–6], and yet time delay can degrade the control performance greatly or even destabilize the system. Time delay has been discussed widely in other related research field like networked control systems; for the consensus of homogeneous MASs with time delay, some reports can also been seen [7–10], but the combination of heterogeneous MASs and time delay still needs some good effort. Indeed, there could be two types of philosophies in the face of time delays, either "passively acceptant" [11, 12], or "actively compensative" [13, 14]. For the former, one can refer to [15] which uses the relatively delayed and periodical intermittent information of neighbor agents, to [16] where MASs without exact knowledge of the network topology, and many others. For the latter, one can refer to, e.g., [17] which uses model predictive control, [18] which discusses the role of the sampling interval, [19, 20] which proposes a decentralized predictive mechanism, [21] which presents event-triggered control, and so forth. One point is worth mentioning that distributed model predictive control is a representative model predictive control schemes. Li *et al.*, for decoupled nonlinear systems, proposed Receding Horizon Control (RHC) algorithm to achieve optimal control performance and handle system constraints more efficiently. The RHC strategy is that the consensus protocol for each agent at each step is generated by solving an optimization problem, in which the arguments for optimization are a sequence of control variables, including the control variable at current time step and the predicted ones in several future steps. For example, [22] reports the preliminary result for the multi-agent systems with linear time invariant dynamics. [23] addresses bounded communication delays by using the robustness constraint and by designing the waiting mechanism, then analyzes the feasibility and stability issues. And [24] which is extended to deal with large-scale nonlinear systems with disturbances and communication delays simultaneously. In these works and some others [25–28], people begin to realize that the predictive based methodology can be an effective way to actively compensate for the delay and therefore has received much attention.

Inspired by the above discussion in this work, we consider the protocol design and output consensus analysis of discrete-time heterogeneous multi-agent systems with delays. A novel distributed protocol is proposed to actively compensate for the constant communication delay with a dynamic output feedback controller. Consensus analysis is also conducted.

The main contributions of this paper can be summarized as follows. First, compared with the works in [11, 12] and [15, 16], we put forward the networked predictive control scheme to compensate delays actively, where an observer is used to predict the forward step based on the obtained time-delay information, then according to the structure of the system model, the information of the next time is predicted until the information of the current time. Second, the problem of state consensus for MASs with heterogeneous dynamics has been investigated in [14] and [29], which have the same dimensions of the state, control input and measured output, respectively. In this paper, we address further the output consensus problem of heterogeneous MASs, where dimensions of the states and control inputs can be different for the dynamics systems except for the dimension of measured outputs. Third, the communication network with a constant time-delay is studied in this manuscript, but the proposed method also applies to MASs with the bounded and time-vary communication delay. Therefore, this manuscript also give a method to deal with bounded and time-vary delay.

The paper is organized as follows. The protocol design and consensus analysis are given in Section II. Numerical examples are provided in Section III. Concluding remarks are drawn in Section IV.

Notations. The sets of nonnegative integers, real number and complex number are denoted by $\mathbb{Z}^+$, $\mathbb{R}$ and $\mathbb{C}$, respectively. $M_{m,n}(\mathbb{F})$ is the set of all m-by-n matrices over a field $\mathbb{F}$ and $M_{n,n}(\mathbb{F})$ is usually abbreviated to $M_n(\mathbb{F})$ if no confusion is caused. A^- and $A^\dagger$ denote the $\{1\}$-inverse and Moone-Penrose inverse of matrix A, respectively. A vector valued function $\mathrm{vec}(\cdot)$ of a matrix is defined as $\mathrm{vec}(A) = [\begin{array}{ccc} A_1^{\mathrm{T}} & \cdots & A_n^{\mathrm{T}} \end{array}]^{\mathrm{T}} \in M_{mn,1}(\mathbb{F})$, where A_k is the k-th column of A, $k = 1, 2, \cdots, n$. A matrix $V \in M_n(\mathbb{C})$ is said to be Schur if $\sigma(V) \subseteq U_0$, where $\sigma(V)$ represents the spectrum of matrix V, and U_0 denotes an open circle of radius 1 centered at 0. The Kronecker product of $A = [a_{ij}] \in M_{m,n}(\mathbb{F})$ and $B = [b_{ij}] \in M_{p,q}(\mathbb{F})$ is denoted by and defined as $A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in M_{mp,nq}(\mathbb{F})$. 0 represents zero matrix with the appropriate dimension, and $\mathbf{1}_N$ denotes a N-dimension column vector with all entries being one. $\mathrm{diag}(\cdot)$ represents a block-diagonal matrix.

2 Protocol design and consensus analysis

2.1 Protocol design

Consider an MAS with N heterogeneous agents, and the dynamics of agent i is described by the following linear discrete-time system,

$$\begin{aligned} x_i(t+1) = &\ A_i x_i(t) + B_i u_i(t), \\ y_i(t) = &\ C_i x_i(t), \\ x_i(t) = &\ \varphi_{xi}(t),\ -2\tau \le t \le 0, \\ u_i(t) = &\ \varphi_{ui}(t),\ -2\tau \le t \le 0, \\ y_i(t) = &\ \varphi_{yi}(t),\ -2\tau \le t \le 0, \end{aligned} \tag{1}$$

where $x_i \in M_{n_i,1}(\mathbb{R})$, $u_i \in M_{m_i,1}(\mathbb{R})$ and $y_i \in M_{l,1}(\mathbb{R})$ are the state, control input and measured output of agent i, respectively; $A_i \in M_{n_i}$, $B_i \in M_{n_i,m_i}$, $C_i \in M_{l,n_i}$ are constant matrices. τ is the transmission delay which is assumed to be constant and bounded. $\varphi_{xi}(\cdot)$, $\varphi_{ui}(\cdot)$ and $\varphi_{yi}(\cdot)$ are the initial state, initial control input and initial output, respectively.

Information exchange between agents in MASs can be modelled by fixed and directed topology. Regarding the above N agents as nodes of a digraph, the communication relationship among agents in (1) can be conveniently represented by a weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with the set of nodes $\mathcal{V} = \{1, 2, \cdots, N\}$, set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a nonnegative weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in M_N(\mathbb{R})$ where $a_{ii} = 0$ for all $i \in \mathcal{V}$. The directed edge $(i, j) \in \mathcal{E}$ means that agent j can receive the information from agent i; if this is so, then agent j is called the sub node, and agent i is the parent node. For agent i, agent j is its neighbor if $(i, j) \in \mathcal{E}$, and the set of all its neighbor nodes is denoted by $N_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$. If there exists a directed path from node i to node j, then node j is said to be reachable from node i. The set of all reachable nodes to node i is denoted by N_i^*. Laplacian matrix $\mathcal{L} = [l_{ij}] \in M_N(\mathbb{R})$ of the weighted digraph $\mathcal{G}$ is defined as $l_{ii} = \sum_{j=1,j\neq i}^{N} a_{ij}, l_{ij} = -a_{ij}, \forall i \neq j$. Obviously, all the row-sums of $\mathcal{L}$ are zero, which implies that $\mathcal{L}$ has always a zero eigenvalue corresponding to the right eigenvector $\mathbf{1}_N$. For more information on graph theory, the reader is referred to [30].

The concept of state consensus can no longer exist for heterogeneous MASs, since even the dimensions of the agents states can be different. On the contrary, output consensus, *i.e.*, $\lim\limits_{t\to\infty} \|y_i(t) - y_j(t)\| = 0,\ \forall\, i, j = 1, 2, \cdots, N$ can be more desirable, and is the topic to be discussed in this work.

In this paper, there exists a constant communication delay among agents. So not current data but delayed data can be received from other agent. If the protocol is designed by using delayed data, the performance and effect of MASs can be unsatisfactory so that prediction based approach is exploited to overcome communication delay actively. Because agent i receives information from agent j ($j \in \{i\} \cup N_i^*$) with time delay τ, in order to overcome the effect of the network delay, based on the output data of agent j up to time $t - \tau$, the state predictions of agent j from time $t - \tau + 1$ to t are constructed as follows,

$$\begin{aligned} \hat{x}_j(t-\tau+1|t-\tau) = &\ A_j \hat{x}_j(t-\tau|t-\tau-1) \\ &+ B_j u_j(t-\tau) + G_j[y_j(t-\tau) \\ &- C_j \hat{x}_j(t-\tau|t-\tau-1)], \end{aligned} \tag{2a}$$

$$\begin{aligned} \hat{x}_j(t-\tau+d|t-\tau) = &\ A_j \hat{x}_j(t-\tau+d-1|t-\tau) \\ &+ B_j u_j(t-\tau+d-1), \\ d = &\ 2, 3, \cdots, \tau,\ j \in \{i\} \cup N_i^*, \end{aligned} \tag{2b}$$

where $\hat{x}_j(t-\tau+1|t-\tau) \in M_{n_j,1}(\mathbb{R})$ and $u_j(t-\tau) \in M_{m_j,1}(\mathbb{R})$ are the one-step ahead state prediction and the input of the observer at time $t-\tau$, respectively. $\hat{x}_j(t-\tau+d|t-\tau) \in M_{n_j,1}(\mathbb{R})$ is the state prediction of agent j at time $t-\tau+d$ on the basis of the information up to time $t-\tau$, and $u_j(t-\tau+d-1) \in M_{m_j,1}(\mathbb{R})$ is the input at time $t-\tau+d-1$, $d = 2, 3, \cdots, \tau$, $j \in \{i\} \cup N_i^*$.

We design the following protocol,

$$\begin{aligned} z_i(t+1) = &\ \hat{A}_i z_i(t) + \hat{B}_i \hat{y}_i(t|t-\tau) + \hat{H}_i \hat{\zeta}_i(t|t-\tau), \\ u_i(t) = &\ \hat{C}_i z_i(t) + \hat{D}_i \hat{y}_i(t|t-\tau) + \hat{F}_i \hat{\zeta}_i(t|t-\tau), \\ z_i(t) = &\ \varphi_{zi}(t), -2\tau \le t \le 0, i \in \mathcal{V} \end{aligned} \tag{3}$$

where $z_i(t) \in M_{\widetilde{n}_i,1}(\mathbb{R})$ is the protocol state, $\hat{y}_i(t|t-\tau) = C_i \hat{x}_i(t|t-\tau)$ is the output prediction at time t, based on the data of agent i up to time $t-\tau$, $\hat{\zeta}_i(t|t-\tau) = \sum_{j\in N_i} a_{ij} \Delta\hat{y}_{ij}(t|t-\tau)$, $\Delta\hat{y}_{ij}(t|t-\tau) = \hat{y}_j(t|t-\tau) - \hat{y}_i(t|t-\tau)$ is the output prediction difference between agent i and agent j, $\varphi_{zi}(t)$ is the initial state of the protocol. $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, $\hat{D}_i$, $\hat{F}_i$ and $\hat{H}_i$ are matrices to be designed, $i = 1, 2, \cdots, N$.

2.2 Consensus analysis

Definition 1. *For discrete-time MASs* (1)*, protocol* (3) *is said to solve the consensus problem if the following conditions hold,*

(1) $\lim\limits_{t\to\infty} \|y_i(t) - y_j(t)\| = 0,\ \forall\, i, j = 1, 2, \cdots, N$;
(2) $\lim\limits_{t\to\infty} \|z_i(t)\| = 0, \forall\, i = 1, 2, \cdots, N$,
(3) $\lim\limits_{t\to\infty} \|e_i(t)\| = 0, \forall\, i = 1, 2, \cdots, N$,

where $e_i(t) = \hat{x}_i(t|t-1) - x_i(t)$ *is the one-step ahead estimate error.*

Let

$$\begin{aligned} &\delta_i(t) = y_i(t) - y_1(t),\ i = 1, 2, \cdots, N, \\ &\delta(t) = [\begin{array}{cccc} \delta_2^{\mathrm{T}}(t) & \delta_3^{\mathrm{T}}(t) & \cdots & \delta_N^{\mathrm{T}}(t) \end{array}]^{\mathrm{T}}, \\ &x(t) = [\begin{array}{cccc} x_1^{\mathrm{T}}(t) & x_2^{\mathrm{T}}(t) & \cdots & x_N^{\mathrm{T}}(t) \end{array}]^{\mathrm{T}}, \\ &y(t) = [\begin{array}{cccc} y_1^{\mathrm{T}}(t) & y_2^{\mathrm{T}}(t) & \cdots & y_N^{\mathrm{T}}(t) \end{array}]^{\mathrm{T}}, \\ &e(t) = [\begin{array}{cccc} e_1^{\mathrm{T}}(t) & e_2^{\mathrm{T}}(t) & \cdots & e_N^{\mathrm{T}}(t) \end{array}]^{\mathrm{T}}. \\ &z(t) = [\begin{array}{cccc} z_1^{\mathrm{T}}(t) & z_2^{\mathrm{T}}(t) & \cdots & z_N^{\mathrm{T}}(t) \end{array}]^{\mathrm{T}}. \end{aligned}$$

From Definition 1, protocol (3) solves the output consensus problem if and only if $\lim\limits_{t\to\infty} \|\delta(t)\| = 0$, $\lim\limits_{t\to\infty} \|z(t)\| = 0$ and $\lim\limits_{t\to\infty} \|e(t)\| = 0$.

For the sake of simplicity, denoted by

$$
\begin{aligned}
R &= \begin{bmatrix} -\mathbf{1}_{N-1} & I_{N-1} \end{bmatrix} \otimes I_l, \\
A_D &= \mathrm{diag}(A_1, A_2, \cdots, A_N), \\
B_D &= \mathrm{diag}(B_1, B_2, \cdots, B_N), \\
C_D &= \mathrm{diag}(C_1, C_2, \cdots, C_N), \\
D_D &= \mathrm{diag}(D_1, D_2, \cdots, D_N), \\
\hat{A}_D &= \mathrm{diag}(\hat{A}_1, \hat{A}_2, \cdots, \hat{A}_N), \\
\hat{C}_D &= \mathrm{diag}(\hat{C}_1, \hat{C}_2, \cdots, \hat{C}_N), \\
\hat{H}_D &= \mathrm{diag}(\hat{H}_1, \hat{H}_2, \cdots, \hat{H}_N), \\
\hat{F}_D &= \mathrm{diag}(\hat{F}_1, \hat{F}_2, \cdots, \hat{F}_N).
\end{aligned}
$$

Assume that (A_i, C_i) is detectable, there exists $G_i \in M_{n_i,l}(\mathbb{R})$ such that $A_i - G_iC_i$ is Schur, $i = 1, 2, \cdots, N$. Take G_i as a gain matrix of observer (2a), $i = 1, 2, \cdots, N$. For agent i, it follows from (2) that the predictive state of agent j at time t is

$$
\begin{aligned}
\hat{x}_j(t|t-\tau) = \; & A_j^{\tau-1}(A_j - G_jC_j)\hat{x}_j(t-\tau|t-\tau-1) \\
& + \sum_{s=1}^{\tau} A_j^{\tau-s}B_ju_j(t-\tau+s-1) \\
& + A_j^{\tau-1}G_jy_j(t-\tau), \; j \in \{i\} \cup N_i^*
\end{aligned} \tag{4}
$$

From (1), the system state can be expressed by

$$
\begin{aligned}
x_i(t) &= A_ix_i(t-1) + B_iu_i(t-1) \\
&= A_i^2x_i(t-2) + A_iB_iu_i(t-2) + B_iu_i(t-1) \\
&= \cdots \\
&= A_i^{\tau}x_i(t-\tau) + \sum_{s=1}^{\tau} A_i^{\tau-s}B_iu_i(t-\tau+s-1)
\end{aligned} \tag{5}
$$

Combining (4) and (5) yields

$$
\begin{aligned}
\hat{x}_j(t|t-\tau) &= x_j(t) + A_j^{\tau-1}e_j(t-\tau+1), \\
\hat{y}_j(t|t-\tau) &= y_j(t) + C_jA_j^{\tau-1}e_j(t-\tau+1), \\
& \quad j \in \{i\} \cup N_i^*
\end{aligned} \tag{6}
$$

Therefore

$$
\hat{\zeta}_i(t|t-\tau) = -\sum_{j=1}^{N} l_{ij}\left[y_j(t) + C_jA_j^{\tau-1}e_j(t-\tau+1)\right] \tag{7}
$$

Substituting (6) and (7) into (3) derives

$$
\begin{aligned}
u_i(t) = \; & \hat{C}_iz_i(t) + \hat{D}_i[y_i(t) + C_iA_i^{\tau-1}e_i(t-\tau+1)] \\
& + \hat{F}_i \sum_{j\in N_i} a_{ij}[\hat{y}_j(t|t-\tau) - \hat{y}_i(t|t-\tau)] \\
= \; & \hat{C}_iz_i(t) + \hat{D}_iy_i(t) - \hat{F}_i \sum_{j=2}^{N} l_{ij}\delta_j(t) \\
& + \hat{D}_iC_iA_i^{\tau-1}e_i(t-\tau+1) \\
& - \hat{F}_i \sum_{j=1}^{N} l_{ij}C_jA_j^{\tau-1}e_j(t-\tau+1)
\end{aligned}
$$

The closed-loop system can then be described as

$$
\begin{aligned}
& x_i(t+1) \\
= \; & A_ix_i(t) + B_iu_i(t) \\
= \; & A_ix_i(t) + B_i\hat{C}_iz_i(t) + B_i\hat{D}_iy_i(t) \\
& - B_i\hat{F}_i(\tilde{l}_i \otimes I_l)\delta(t) + B_i\hat{D}_iC_iA_i^{\tau-1}e_i(t-\tau+1) \\
& - B_i\hat{F}_i(l_i \otimes I_l)C_DA_D^{\tau-1}e(t-\tau+1)
\end{aligned} \tag{8}
$$

Then

$$
\begin{aligned}
& y_i(t+1) \\
= \; & C_i(A_i + B_i\hat{D}_iC_i)x_i(t) + C_iB_i\hat{C}_iz_i(t) \\
& - C_iB_i\hat{F}_i(\tilde{l}_i \otimes I_l)\delta(t) + C_iB_i\hat{D}_iC_iA_i^{\tau-1}e_i(t-\tau+1) \\
& - C_iB_i\hat{F}_i(l_i \otimes I_l)C_DA_D^{\tau-1}e(t-\tau+1)
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
& z_i(t+1) \\
= \; & \hat{A}_iz_i(t) + \hat{B}_i\hat{y}_i(t|t-\tau) + \hat{H}_i\hat{\zeta}_i(t|t-\tau) \\
= \; & \hat{A}_iz_i(t) + \hat{B}_i[y_i(t) + C_iA_i^{\tau-1}e_i(t-\tau+1)] \\
& - \hat{H}_i \sum_{j=2}^{N} l_{ij}\delta_j(t) - \hat{H}_i \sum_{j=1}^{N} l_{ij}C_jA_j^{\tau-1}e_j(t-\tau+1)
\end{aligned} \tag{10}
$$

where $\tilde{l}_i = l_i\begin{bmatrix} 0 & I_{N-1} \end{bmatrix}^{\mathrm{T}}$, l_i is the i-th row of Laplacian matrix $\mathcal{L}$.

Theorem 1. *Consider discrete-time MASs (1) with fixed and directed topology. Protocol (3) solves the output consensus problem if the following conditions hold,*

(a$_1$) (A_i, C_i) *is detectable,* $i \in \mathcal{V}$*;*
(a$_2$) $\mathrm{rank}(C_i^{\mathrm{T}} \otimes C_iB_i) = \mathrm{rank}([C_i^{\mathrm{T}} \otimes C_iB_i \;\; \mathrm{vec}(C_iA_i)])$, $i \in \mathcal{V}$*;*
(a$_3$) $\hat{B}_iC_i = 0$, $i \in \mathcal{V}$*;*
(a$_4$) *Matrix* Γ *is Schur,*

where

$$
\Gamma = \begin{bmatrix} RC_DB_D & 0 \\ 0 & I_{\tilde{n}} \end{bmatrix} \begin{bmatrix} \hat{F}_D & \hat{C}_D \\ \hat{H}_D & \hat{A}_D \end{bmatrix} \begin{bmatrix} -(\mathcal{L}_2 \otimes I_l) & 0 \\ 0 & I_{\tilde{n}} \end{bmatrix} \tag{11}
$$

$\tilde{n} = \sum_{i=1}^{N} \tilde{n}_i$, $\mathcal{L}_2 = \mathcal{L}\begin{bmatrix} 0 & I_{N-1} \end{bmatrix}^{\mathrm{T}}$ *and* $\mathcal{L}$ *is the Laplacian matrix of digraph* $\mathcal{G}$.

Proof: It follows from condition (a$_2$) that there exists $\hat{D}_i \in M_{m_i,l}(\mathbb{R})$, satisfying $C_i(A_i + B_i\hat{D}_iC_i) = 0$, *i.e.*, $C_iB_i\hat{D}_iC_i = -C_iA_i$, $i \in \mathcal{V}$.

From (a$_3$), take $\hat{B}_i$ such that $\hat{B}_iC_i = 0$, $i \in \mathcal{V}$. (9) and (10) is reduced to

$$
\begin{aligned}
y_i(t+1) = \; & C_iB_i\hat{C}_iz_i(t) - C_iB_i\hat{F}_i(\tilde{l}_i \otimes I_l)\delta(t) \\
& - C_iA_i^{\tau}e_i(t-\tau+1) \\
& - C_iB_i\hat{F}_i(l_i \otimes I_l)C_DA_D^{\tau-1}e(t-\tau+1),
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
z_i(t+1) = \; & \hat{A}_iz_i(t) - \hat{H}_i(\tilde{l}_i \otimes I_l)\delta(t) \\
& - \hat{H}_i(l_i \otimes I_l)C_DA_D^{\tau-1}e(t-\tau+1)
\end{aligned}
$$

Then, the compact form of closed-loop systems is as follows

$$
\begin{aligned}
y(t+1) = \; & C_DB_D\hat{C}_Dz(t) - C_DB_D\hat{F}_D(\mathcal{L}_2 \otimes I_l)\delta(t) \\
& - [C_DA_D^{\tau} + C_DB_D\hat{F}_D(\mathcal{L} \otimes I_l)C_DA_D^{\tau-1}] \\
& e(t-\tau+1), \\
z(t+1) = \; & \hat{A}_Dz(t) - \hat{H}_D(\mathcal{L}_2 \otimes I_l)\delta(t) \\
& - \hat{H}_D(\mathcal{L} \otimes I_l)C_DA_D^{\tau-1}e(t-\tau+1)
\end{aligned}
$$

Because $\delta(t) = Ry(t)$, the generalized closed-loop system can be described as

$$
\xi(t+1) = \Omega\xi(t) \tag{13}
$$

where

$$
\xi(t) = \begin{bmatrix} \delta^{\mathrm{T}}(t) & z^{\mathrm{T}}(t) & e^{\mathrm{T}}(t-\tau+1) \end{bmatrix}^{\mathrm{T}},
$$

$$
\Omega = \begin{bmatrix} \Gamma & \Omega_1 \\ 0 & A_D - G_DC_D \end{bmatrix},
$$

$$\Omega_1 = \left[\begin{array}{cc} \Omega_{11}^{\mathrm{T}}(t) & \Omega_{12}^{\mathrm{T}}(t) \end{array} \right]^{\mathrm{T}},$$

$$\begin{array}{rcl} \Omega_{11} & = & RC_D[B_D\hat{F}_D(\mathcal{L}\otimes I_l)C_D - A_D]A_D^{\tau-1}, \\ \Omega_{12} & = & -\hat{H}_D(\mathcal{L}\otimes I_l))C_D A_D^{\tau-1}, \end{array}$$

It follows from condition (a_4) that Γ is Schur, so system (13) is asymptotically stable. Hence, from Definition 1, protocol (3) solves the output consensus problem.The proof is completed. □

Lemma 1. *[31] Let $A \in M_{m,n}(\mathbb{R})$, $B \in M_{n,m}(\mathbb{R})$. Then matrices AB and BA have the same nonzero eigenvalues.*

Corollary 1. *Consider discrete-time MASs (1) with fixed and directed topology. Protocol (3) solves the output consensus problem, if (a_1), (a_2) and (a_3) in Theorem 1 hold and matrix Γ_1 is Schur, where Γ_1 is defined as*

$$\Gamma_1 = \left[\begin{array}{cc} \hat{F}_D & \hat{C}_D \\ \hat{H}_D & \hat{A}_D \end{array} \right] \left[\begin{array}{cc} -(\mathcal{L}\otimes I_l)C_D B_D & 0 \\ 0 & I_{\widetilde{n}} \end{array} \right]$$

where $\mathcal{L}$ is the Laplacian matrix of digraph $\mathcal{G}$.

Proof: From Lemma 1, Γ in (11) is Schur if and only if

$$\Gamma_1 = \left[\begin{array}{cc} \hat{F}_D & \hat{C}_D \\ \hat{H}_D & \hat{A}_D \end{array} \right] \left[\begin{array}{cc} -(\mathcal{L}_2\otimes I_l) & 0 \\ 0 & I_{\widetilde{n}} \end{array} \right] \left[\begin{array}{cc} RC_D B_D & 0 \\ 0 & I_{\widetilde{n}} \end{array} \right]$$

is Schur.

Note that

$$0 = \mathcal{L}\mathbf{1}_N = \left[\begin{array}{c} \mathcal{L}_{11} + \mathcal{L}_{12}\mathbf{1}_{N-1} \\ \mathcal{L}_{21} + \mathcal{L}_{22}\mathbf{1}_{N-1} \end{array} \right]$$

So $\mathcal{L}_{11} = -\mathcal{L}_{12}\mathbf{1}_{N-1}$ and $\mathcal{L}_{21} = -\mathcal{L}_{22}\mathbf{1}_{N-1}$.

Then

$$\begin{array}{rcl} (\mathcal{L}_2\otimes I_l)R = & & (\left[\begin{array}{c} \mathcal{L}_{12} \\ \mathcal{L}_{22} \end{array} \right]\otimes I_l)(\left[\begin{array}{cc} -\mathbf{1}_{N-1} & I_{N-1} \end{array} \right]\otimes I_l) \\ & = & \left[\begin{array}{cc} -\mathcal{L}_{12}\mathbf{1}_{N-1} & \mathcal{L}_{12} \\ -\mathcal{L}_{22}\mathbf{1}_{N-1} & \mathcal{L}_{22} \end{array} \right]\otimes I_l \\ & = & \left[\begin{array}{cc} -\mathcal{L}_{11} & \mathcal{L}_{12} \\ -\mathcal{L}_{21} & \mathcal{L}_{22} \end{array} \right]\otimes I_l \\ & = & \mathcal{L}\otimes I_l \end{array}$$

Hence,

$$\begin{array}{rcl} \Gamma_1 = & & \left[\begin{array}{cc} \hat{F}_D & \hat{C}_D \\ \hat{H}_D & \hat{A}_D \end{array} \right] \left[\begin{array}{cc} -(\mathcal{L}_2\otimes I_l)RC_D B_D & 0 \\ 0 & I_{\widetilde{n}} \end{array} \right] \\ & = & \left[\begin{array}{cc} \hat{F}_D & \hat{C}_D \\ \hat{H}_D & \hat{A}_D \end{array} \right] \left[\begin{array}{cc} -(\mathcal{L}\otimes I_l)C_D B_D & 0 \\ 0 & I_{\widetilde{n}} \end{array} \right]. \end{array}$$

The proof is completed. □

Theorem 2. *Consider discrete-time MASs (1) with fixed and directed topology. Protocol (3) solves the output consensus problem if*

(i) (a_1) *and* (a_2) *in Theorem 1 hold;*
(ii) $\hat{B}_i C_1 = 0$, $i \in \mathcal{V}$;
(iii) *Matrix $\widetilde{\Gamma}$ is Schur,*

where

$$\widetilde{\Gamma} = \left[\begin{array}{cc} RC_D B_D & 0 \\ 0 & I_{\widetilde{n}} \end{array} \right] \left[\begin{array}{cc} -\hat{F}_D(\mathcal{L}_2\otimes I_l) & \hat{C}_D \\ \hat{B}_D - \hat{H}_D(\mathcal{L}_2\otimes I_l) & \hat{A}_D \end{array} \right],$$

$\mathcal{L}_2 = \mathcal{L}[\begin{array}{cc} 0 & I_{N-1} \end{array}]^{\mathrm{T}}$ and $\mathcal{L}$ is Laplacian matrix of digraph $\mathcal{G}$.

Proof: From (10) and condition (ii),

$$\begin{array}{rl} & z_i(t+1) \\ = & \hat{A}_i z_i(t) + \hat{B}_i[\delta_i(t) + y_1(t) + C_i A_i^{\tau-1} e_i(t-\tau+1)] \\ & -\hat{H}_i(\widetilde{l}_i\otimes I_l)\delta(t) - \hat{H}_i(l_i\otimes I_l)C_D A_D^{\tau-1} e(t-\tau+1) \\ = & \hat{A}_i z_i(t) + \hat{B}_i\delta_i(t) + \hat{B}_i C_i A_i^{\tau-1} e_i(t-\tau+1) \\ & -\hat{H}_i(\widetilde{l}_i\otimes I_l)\delta(t) - \hat{H}_i(l_i\otimes I_l)C_D A_D^{\tau-1} e(t-\tau+1) \end{array} \quad (14)$$

Combining (12) and (14) yields generalized closed-loop system as follows:

$$\xi(t+1) = \widetilde{\Omega}\xi(t)$$

where

$$\widetilde{\Omega} = \left[\begin{array}{cc} \widetilde{\Gamma} & \widetilde{\Omega}_1 \\ 0 & A_D - G_D C_D \end{array} \right],$$

$$\widetilde{\Omega}_1 = \left[\begin{array}{cc} \Omega_{11}^{\mathrm{T}}(t) & \widetilde{\Omega}_{12}^{\mathrm{T}}(t) \end{array} \right]^{\mathrm{T}},$$

$$\widetilde{\Omega}_{12} \quad = \quad [\hat{B}_D - \hat{H}_D(\mathcal{L}\otimes I_l))]C_D A_D^{\tau-1},$$

Ω_{11} is defined as Theorem 1. It follows from condition (i) and (ii) that protocol (3) solves the consensus problem. The proof is completed. □

Corollary 2. *Consider discrete-time MASs (1) with fixed and directed topology. Protocol (3) solves the output consensus problem, if (i) and (ii) in Theorem 2 hold and matrix $\widetilde{\Gamma}_1$ is Schur, where $\widetilde{\Gamma}_1$ is defined as*

$$\widetilde{\Gamma}_1 = \left[\begin{array}{cc} -\hat{F}_D(\mathcal{L}\otimes I_l) & \hat{C}_D \\ \left[\hat{B}_D R - \hat{H}_D(\mathcal{L}\otimes I_l)\right] C_D B_D & \hat{A}_D \end{array} \right]$$

Proof: The proof is similar to Corollary 1 and can be omitted. □

Remark 1. *When the network has a bounded and time-varying delay $\tau_m \leq \tau(t) \leq \tau_M$ at time t, agent i receives information from agent $j (j \in \{i\} \cup N_i^*)$ with time delay $\tau(t)$, where τ_m and τ_M are known positive integers. According to Wang et al. [32] [33], the dwell-time approach can be used to handle the bounded and time-varying delay. When the network delay $\tau(t) \leq \tau_M$, data in the network are compelled to dwell such that the time delay achieves the upper bound. Then, the time-varying network delay is transformed into constant. In the case of bounded and time-vary delay, the conclusion is still true. Therefore, it is assumed that the network delay is constant in this article. Although the dwell-time approach is slightly conservative, it provides a method of investigating the time-varying delay when it is difficult to directly deal with it.*

Remark 2. *Consider the special case where $n_1 = n_2 = \cdots = n_N = n$, $m_1 = m_2 = \cdots = m_N = m$, and $\widetilde{n}_1 = \widetilde{n}_2 = \cdots = \widetilde{n}_N = \widetilde{n}$. If the condition $\mathrm{rank}(C_{row}R^T) < \mathrm{rank}(C_{row})$, where $C_{row} = [\begin{array}{cccc} C_1 & C_2 & \cdots & C_N \end{array}]$ and $R = [\begin{array}{cc} -\mathbf{1}_{N-1} & I_{N-1} \end{array}]\otimes I_l$ holds, it follows from Lemma 1 in [14] that there exists a non-zero matrice $H \in M_{\widetilde{n},n}(\mathbb{R})$, such that $\mathrm{rank}(C_{row}) = \mathrm{rank}([C_{row}^{\mathrm{T}}\ \mathbf{1}_N \otimes H^{\mathrm{T}}])$. Then the matrix equation*

$$YC_{row} = \mathbf{1}_N^{\mathrm{T}} \otimes H \quad (15)$$

has a solution, and the general solution is $Y = (\mathbf{1}_N^{\mathrm{T}}\otimes H)C_{row}^{-} + Z(I_l - C_{row}C_{row}^{-})$, where $Z \in M_{q,l}(R)$ is arbitrary. In particular, $Y = (\mathbf{1}_N^{\mathrm{T}}\otimes H)C_{row}^{\dagger}$ is a special solution of (15), so $(\mathbf{1}_N^{\mathrm{T}}\otimes H)C_{row}^{\dagger}C_{row} = \mathbf{1}_N^{\mathrm{T}}\otimes H$. It is obvious that $(\mathbf{1}_N^{\mathrm{T}}\otimes H)C_{row}^{\dagger}C_i = H$. The coefficient matrices $\hat{H}_i$ in protocol (3) can be constructed as $\hat{H}_i = (\mathbf{1}_N^{\mathrm{T}}\otimes H)C_{row}^{\dagger}$.

In particular, when communication network has not time delays, protocol (3) is reduced to

$$\begin{aligned} z_i(t+1) = & \hat{A}_i z_i(t) + \hat{B}_i y_i(t) + \hat{H}_i \zeta_i(t), \\ u_i(t) = & \hat{C}_i z_i(t) + \hat{D}_i y_i(t) + \hat{F}_i \zeta_i(t) \end{aligned} \tag{16}$$

where $\zeta_i(t) = \sum_{j\in N_i} a_{ij}\Delta y_{ij}(t)$, $\Delta y_{ij}(t) = y_j(t) - y_i(t)$ is the output difference between agent i and agent j.

For discrete-time MASs (1) without network delay, protocol (16) is said to solve the output consensus problem if $\lim_{t\to\infty} \|y_i(t) - y_j(t)\| = 0$ and $\lim_{t\to\infty} z_i(t) = 0$, $i, j \in \mathcal{V}$. Similar to the above analysis, the augmented system with (16) is $\xi(t+1) = \Gamma\xi(t)$, where $\xi(t) = [\ \delta^{\mathrm{T}}(t) \quad z^{\mathrm{T}}(t)\ ]^{\mathrm{T}}$ and Γ is still defined as (11). Similar to the above proof of Theorem 1, the following result can be readily obtained.

Corollary 3. *Consider discrete-time MASs (1) with $\tau = 0$ and fixed and directed topology. Protocol (16) solves the consensus problem, if (a2), (a3) and ((a4) in Theorem 1 hold.*

3 Simulation examples

Example 1. *Consider discrete-time MASs (1) with $\tau = 4$ and four agents indexed by 1, 2, 3 and 4, respectively. The dynamics of agent are described by equation(1), where*

$$A_1 = \begin{bmatrix} 1.5 & 0 & 1 \\ -3 & 0 & -2 \\ 1 & 0 & 0.5 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 & 0.5 & 1 \\ -2 & -1 & -2 \\ 1 & 1 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.2 & -0.2 & 0.2 \\ 0 & 0 & 0 \\ -0.8 & 3.2 & 3.2 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 2 & -3 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1.2 & 0.6 & 0.6 \\ 0.9 & 0.6 & 3 \\ 1.2 & -0.6 & -0.6 \end{bmatrix},\ B_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 1 \\ -1 & -1 & -1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0.6 & 0.3 & 0.3 \\ 0.6 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.3 \end{bmatrix},\ B_4 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^{\mathrm{T}},\ C_2 = \begin{bmatrix} 2 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}^{\mathrm{T}},$$

$$C_3 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 2 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}},C_4 = \begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & -1 & -3 & 2 \\ 1 & -1 & 2 & 2 \end{bmatrix}^{\mathrm{T}}.$$

The interconnection among the four agents is described by $\mathcal{G}$ in Fig. 1 and the adjacent matrix

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

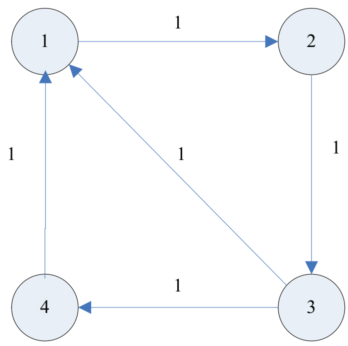

Fig. 1: Fixed topology.

Because $\mathrm{rank}(C_i^{\mathrm{T}} \otimes B_i) = \mathrm{rank}([C_i^{\mathrm{T}} \otimes B_i \mathrm{vec}(C_i A_i)])$, *there exist matrices*

$$\hat{D}_1 = \begin{bmatrix} -0.9444 & -0.2222 & 0.2778 & -0.0556 \\ 0.4444 & -0.7778 & 0.2222 & -0.4444 \\ -0.7778 & -0.8889 & 0.1111 & -1.2222 \end{bmatrix},$$

$$\hat{D}_2 = \begin{bmatrix} -0.1933 & -0.7933 & 0.5200 & -0.2800 \\ -0.2907 & 0.7733 & -0.1920 & 0.8400 \\ 0.5347 & -0.3133 & 0.3440 & -2.0800 \end{bmatrix},$$

$$\hat{D}_3 = \begin{bmatrix} -1.1792 & 0.0792 & 0.2583 & 2.5750 \\ 0.5583 & 0.0417 & 0.4833 & -0.1500 \\ -0.1792 & 0.0792 & 0.2583 & 0.5750 \end{bmatrix},$$

$$\hat{D}_4 = \begin{bmatrix} 0.0667 & 0.4133 & -0.1000 & -0.8267 \\ 0.0833 & -1.1233 & 0.1000 & 0.2467 \\ -0.9833 & -0.5367 & 0.2000 & -0.9267 \end{bmatrix}.$$

satisfying $C_i(A_i + B_i\hat{D}_iC_i) = 0,\ i = 1, 2, 3, 4$. When (A_i, C_i) is detectable, for an arbitrary positive definite matrix Q_i, discrete-time algebraic Riccati equation

$$A_iP_iA_i^T - P_i - A_iP_iC_i^T(R + C_iP_iC_i^T)^{-1}C_iP_iA_i^T + Q_i = 0$$

has an unique solution $P_i > 0$ such that $A_i - G_iC_i$ is Schur, where $G_i = A_iP_iC_i^T(R + C_iP_iC_i^T)^{-1}, i = 1, 2, 3, 4$. Then, feedback gain matrices can be obtained as follows

$$G_1 = \begin{bmatrix} -0.5658 & 0.9798 & 0.1109 & 0.5658 \\ -1.1315 & 1.9595 & 0.2218 & 1.1315 \\ -0.7483 & 0.6770 & -0.0185 & 0.7483 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} -0.0435 & -0.0831 & 0.0962 & 0.0764 \\ 0.0058 & 0.2086 & -0.0004 & 0.1011 \\ 0.1389 & 0.0323 & -0.0383 & -0.0916 \end{bmatrix},$$

$$G_3 = \begin{bmatrix} 0.1382 & 0.1574 & 0.0192 & 0.0569 \\ 0.9046 & 1.2447 & 0.3401 & -0.4876 \\ 0.1992 & 0.2060 & 0.0068 & 0.1982 \end{bmatrix},$$

$$G_4 = \begin{bmatrix} -0.3522 & -0.0308 & 0.1401 & 0.0616 \\ -0.0215 & -0.0786 & -0.0453 & 0.1573 \\ 0.0138 & -0.0570 & -0.0027 & 0.1141 \end{bmatrix}.$$

by choosing $Q_1 = Q_2 = Q_3 = diag(1, 1, 2)$ *and* $Q_4 = diag(1, 1, 4)$. *The solutions of Riccati equation is obtained as follows*

$$P_1 = \begin{bmatrix} 2.2843 & -2.5685 & 0.8379 \\ -2.5685 & 6.1371 & -1.6758 \\ 0.8379 & -1.6758 & 2.5604 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 1.0333 & 0 & 0.0289 \\ 0 & 1 & 0 \\ 0.0289 & 0 & 24.2120 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 6.0145 & 5.8239 & -5.0145 \\ 5.8239 & 10.6571 & -5.8239 \\ -5.0145 & -5.8239 & 7.0145 \end{bmatrix},$$
$$P_4 = \begin{bmatrix} 1.0912 & 0.0912 & 0.0145 \\ 0.0912 & 1.0912 & 0.0145 \\ 0.0145 & 0.0145 & 4.0379 \end{bmatrix}.$$

Take $\hat{H}_i = \begin{bmatrix} 0 & -0.7071 & 0 & -0.7071 \\ 0 & 0.5657 & 0 & 0.5657 \\ 0 & -0.3536 & 0 & -0.3536 \end{bmatrix}$, $i = 1, 2, 3, 4$. *In order to satisfy* $\hat{B}_i C_i = 0$, *we let*

$$\hat{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ -1 & 0 & 0 & -1 \end{bmatrix},$$
$$\hat{B}_2 = \begin{bmatrix} 1 & -1 & -2 & 2 \\ 0.5 & -0.5 & -1 & 1 \\ 0.2 & -0.2 & -0.4 & 0.4 \end{bmatrix},$$
$$\hat{B}_3 = \begin{bmatrix} -1 & 1 & -1 & 0 \\ 2 & -2 & 2 & 0 \\ -0.8 & 0.8 & -0.8 & 0 \end{bmatrix},$$
$$\hat{B}_4 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & -4 & 0 & -2 \\ 0 & 0.4 & 0 & 0.2 \end{bmatrix}.$$

By solving $\Gamma^T P \Gamma - P < 0$, *it is obtained that*

$$\begin{aligned} \hat{A}_1 = {} & diag(2, 2, 5), \hat{A}_2 = diag(5, 2, 3), \\ \hat{A}_3 = {} & diag(1, 8, 8), \hat{A}_4 = diag(4, 1, 5), \\ \hat{C}_1 = {} & \begin{bmatrix} 2 & 1 & 3 \\ -2 & -1 & -3 \\ -1 & -0.5 & -1.5 \end{bmatrix}, \\ \hat{C}_2 = {} & \begin{bmatrix} 0.5 & 1 & 1.5 \\ 0.1 & 0.2 & 0.3 \\ -0.7 & -1.4 & -2.1 \end{bmatrix}, \\ \hat{C}_3 = {} & \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ -1 & -2 & -2 \end{bmatrix}, \\ \hat{C}_4 = {} & \begin{bmatrix} -2 & 1 & 2.6 \\ 2 & -1 & -2.6 \\ -2 & 1 & 2.6 \end{bmatrix}. \end{aligned}$$

There exists matrices $\hat{F}_i$ *to verify that* $-RC_D B_D F_D(\mathcal{L}_2 \otimes I_l)$ *is Schur, where* $\hat{F}_i = \begin{bmatrix} 0 & -0.2475 & 0 & -0.2475 \\ 0 & 0.1273 & 0 & 0.1273 \\ 0 & 0.4243 & 0 & 0.4243 \end{bmatrix}$, $i = 1, 2, 3, 4$. *Hence, protocol* (3) *solves the output consensus problem by Theorem* 1.

For $\tau = 4$, *the initial conditions of discrete-time MASs* (1) *are chosen as*

$$\begin{aligned} & x_1(0) = \begin{bmatrix} -1\,2\,2\,-1 \end{bmatrix}^{\mathrm{T}}, x_2(0) = \begin{bmatrix} 3\,-3\,4\,-4 \end{bmatrix}^{\mathrm{T}}, \\ & x_3(0) = \begin{bmatrix} 5\,6\,-6\,-5 \end{bmatrix}^{\mathrm{T}}, x_4(0) = \begin{bmatrix} 7\,-7\,8\,1 \end{bmatrix}^{\mathrm{T}}, \\ & z_1(0) = \begin{bmatrix} 1\,0\,2 \end{bmatrix}^{\mathrm{T}}, z_2(0) = \begin{bmatrix} 0\,-2\,0 \end{bmatrix}^{\mathrm{T}}, \\ & z_3(0) = \begin{bmatrix} -1\,0\,3 \end{bmatrix}^{\mathrm{T}}, z_4(0) = \begin{bmatrix} 0\,-3\,0 \end{bmatrix}^{\mathrm{T}}, \\ & e_1(0) = \begin{bmatrix} 0.1\,0.3\,0.5 \end{bmatrix}^{\mathrm{T}}, e_2(0) = \begin{bmatrix} 0.3\,0.2\,0.4 \end{bmatrix}^{\mathrm{T}}, \\ & e_3(0) = \begin{bmatrix} -0.2\,0.4\,-0.1 \end{bmatrix}^{\mathrm{T}}, e_4(0) = \begin{bmatrix} 0.1\,0.5\,1 \end{bmatrix}^{\mathrm{T}}, \\ & e_1(-1) = \begin{bmatrix} 0.4\,0.2\,0.6 \end{bmatrix}^{\mathrm{T}}, e_2(-1) = \begin{bmatrix} 0.1\,0.4\,0.3 \end{bmatrix}^{\mathrm{T}}, \\ & e_3(-1) = \begin{bmatrix} -0.2\,-1\,0.1 \end{bmatrix}^{\mathrm{T}}, \\ & e_4(-1) = \begin{bmatrix} 0.3\,0.4\,0.2 \end{bmatrix}^{\mathrm{T}}, \\ & e_1(-2) = \begin{bmatrix} 1\,0.1\,0.3 \end{bmatrix}^{\mathrm{T}}, e_2(-2) = \begin{bmatrix} 0.1\,1\,0.4 \end{bmatrix}^{\mathrm{T}}, \\ & e_3(-2) = \begin{bmatrix} 0.8\,0.5\,0.2 \end{bmatrix}^{\mathrm{T}}, \\ & e_4(-2) = \begin{bmatrix} 0.6\,0.8\,-0.1 \end{bmatrix}^{\mathrm{T}}, \\ & e_1(-3) = \begin{bmatrix} 1\,0.9\,0.2 \end{bmatrix}^{\mathrm{T}}, e_2(-3) = \begin{bmatrix} 0.1\,0.7\,1 \end{bmatrix}^{\mathrm{T}}, \\ & e_3(-3) = \begin{bmatrix} 0.3\,-0.2\,0.1 \end{bmatrix}^{\mathrm{T}}, \\ & e_4(-3) = \begin{bmatrix} 0.5\,0.1\,-0.3 \end{bmatrix}^{\mathrm{T}}. \end{aligned}$$

Sample output trajectories of the agents, trajectories of protocol state and estimate error trajectories are shown in Fig. 2- 4.

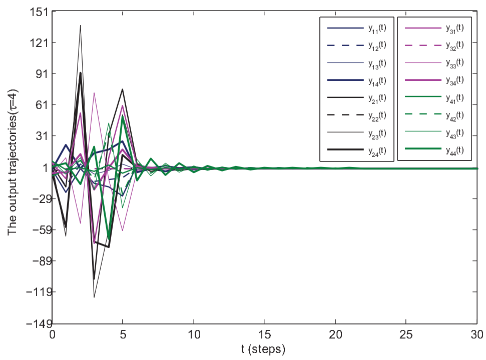

Fig. 2: The output trajectories $y_i(t), i = 1, 2, 3, 4$ ($\tau = 4$).

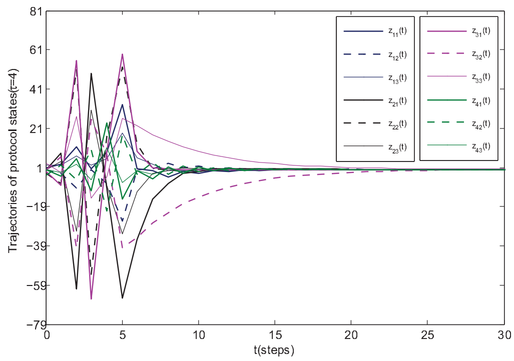

Fig. 3: Trajectories of protocol states $z_i(t), i = 1, 2, 3, 4$ ($\tau = 4$).

Example 2. *Consider the output consensus of discrete-time MASs* (1) *with four agents indexed by* 1, 2, 3 *and* 4, *respectively. The*

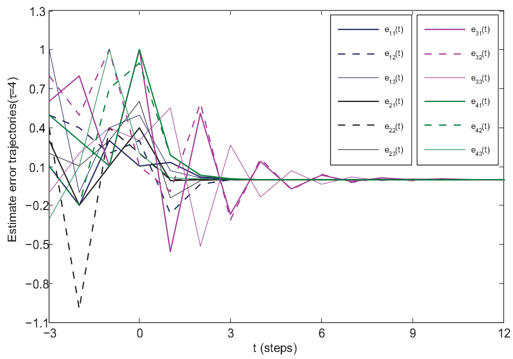

Fig. 4: Estimate error trajectories $e_i(t), i = 1, 2, 3, 4$ ($\tau = 4$).

dynamics of agents are described by equation (1)*, where*

$$A_1 = \begin{bmatrix} 1.2 & 0 \\ -2.4 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ 0.2 & 1.8 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & 0.5 \\ 1 & -0.5 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0.45 & 0.9 \\ 0.9 & 1.8 \end{bmatrix}, B_4 = \begin{bmatrix} 1 & 0.5 \\ 2 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, C_2 = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}^{\mathrm{T}},$$

$$C_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}, C_4 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}^{\mathrm{T}}.$$

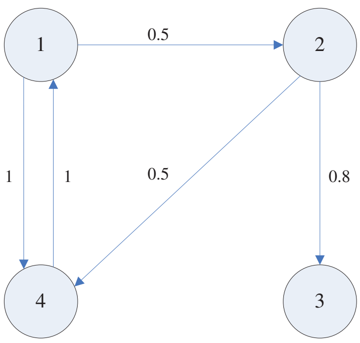

Fig. 5: Fixed topology.

The interconnection among the four agents is described by $\mathcal{G}$ in Fig. 5 *and the adjacent matrix*

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 1 & 0.5 & 0 & 0 \end{bmatrix}.$$

Conditions with no communication delay is the same as conditions with communication delay to achieve the output consensus. By choosing $Q_1 = Q_2 = Q_3 = diag(1, 3)$ and $Q_4 = diag(1, 6)$, and solutions of Riccati equation(1) *the gain matrices are obtained using Matlab,*

$$P_1 = \begin{bmatrix} 1.8548 & -1.7097 \\ -1.7097 & 6.4194 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 3.4125 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1.6024 & -0.6024 \\ -0.6024 & 3.6024 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 1.2397 & 0.4795 \\ 0.4795 & 6.9590 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0.1124 & 0.7124 & 0 \\ -0.2247 & -1.4247 & 0 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0.4789 & -0.0930 & 0.3860 \end{bmatrix},$$

$$G_3 = \begin{bmatrix} -0.3926 & -0.3926 & 0.4195 \\ 0.3926 & 0.3926 & -0.4195 \end{bmatrix},$$

$$G_4 = \begin{bmatrix} 0.1669 & -0.1989 & 0.3339 \\ 0.3339 & -0.3978 & 0.6677 \end{bmatrix}.$$

It can be chosen that $\hat{A}_1 = diag(0.2, 0.5)$, $\hat{A}_2 = diag(0.5, 0.2)$, $\hat{A}_3 = diag(0.1, 0.8)$, $\hat{A}_4 = diag(0.4, 0.5)$, $\hat{B}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $\hat{B}_2 = \begin{bmatrix} 1.5 & 1.5 & -1.5 \\ 0.5 & 0.5 & -0.5 \end{bmatrix}$, $\hat{B}_3 = \begin{bmatrix} 1.2 & -1.2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$, $\hat{B}_4 = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \end{bmatrix}$, $\hat{C}_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, $\hat{C}_2 = \begin{bmatrix} -0.5 & -0.5 \\ 1 & 1 \end{bmatrix}$, $\hat{C}_3 = \begin{bmatrix} -0.8 & -0.8 \\ 0.8 & 0.8 \end{bmatrix}$, $\hat{C}_4 = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$, $\hat{H}_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.8 & 0 \end{bmatrix}$, $\hat{F}_i = \begin{bmatrix} 0 & 0.35 & 0 \\ 0 & -0.6 & 0 \end{bmatrix}$, $i = 1, 2, 3, 4$.

When $\tau = 0$ and $\tau = 2$,the initial state and initial protocol states of discrete-time MASs (1) *are both*

$$\begin{aligned} x_1(0) &= \begin{bmatrix} 0 & 3 & 2 \end{bmatrix}^{\mathrm{T}}, x_2(0) = \begin{bmatrix} 1 & -1 & 5 \end{bmatrix}^{\mathrm{T}}, \\ x_3(0) &= \begin{bmatrix} 6 & -2 & -10 \end{bmatrix}^{\mathrm{T}}, x_4(0) = \begin{bmatrix} 8 & -3 & 1 \end{bmatrix}^{\mathrm{T}}, \\ z_1(0) &= \begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}, z_2(0) = \begin{bmatrix} -3 & 3 \end{bmatrix}^{\mathrm{T}}, \\ z_3(0) &= \begin{bmatrix} -4 & 4 \end{bmatrix}^{\mathrm{T}}, z_4(0) = \begin{bmatrix} 5 & -5 \end{bmatrix}^{\mathrm{T}}. \end{aligned}$$

When $\tau = 2$, the initial estimate error is

$$\begin{aligned} e_1(0) &= \begin{bmatrix} 0.4 & -0.5 \end{bmatrix}^{\mathrm{T}}, e_2(0) = \begin{bmatrix} 0.5 & 0.1 \end{bmatrix}^{\mathrm{T}}, \\ e_1(-1) &= \begin{bmatrix} 0.3 & 0.2 \end{bmatrix}^{\mathrm{T}}, e_2(-1) = \begin{bmatrix} 0.5 & -0.4 \end{bmatrix}^{\mathrm{T}}, \\ e_1(-2) &= \begin{bmatrix} 1 & 0.2 \end{bmatrix}^{\mathrm{T}}, e_2(-2) = \begin{bmatrix} -0.5 & 0.5 \end{bmatrix}^{\mathrm{T}}, \\ e_1(-3) &= \begin{bmatrix} -0.4 & 0.5 \end{bmatrix}^{\mathrm{T}}, e_2(-3) = \begin{bmatrix} 0.2 & -0.8 \end{bmatrix}^{\mathrm{T}}. \end{aligned}$$

Compared with $\tau = 0$ and $\tau = 2$, the output trajectories of four agents and trajectories of protocol state are shown in Fig. 6*–Fig.* 8*, respectively. When communication delay $\tau = 2$, the estimate error trajectory is shown in Fig.* 9*. In the case of no communication delay, the states of system* (1) *take 11 steps to reach the output consensus. In the case of $\tau = 2$ and $\tau = 8$, the states of system* (1) *both achieve the consensus after 12 steps. These results show that performance of discrete-time MASs with the networked predictive control method to compensate for the delay is very close to systems without any communication delay.*

4 Conclusions

The output consensus of discrete-time heterogeneous MASs is investigated in the presence of constant communication delay. A novel

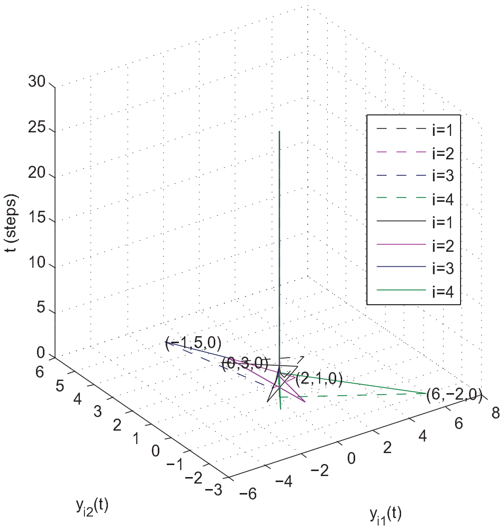

Fig. 6: The output trajectories $y_{i1}(t), y_{i2}(t), i = 1, 2, 3, 4$ ($\tau = 0, \tau = 2$) .

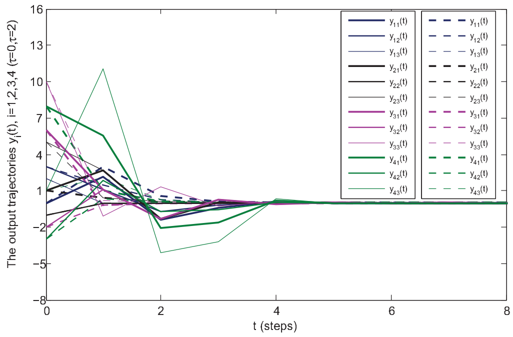

Fig. 7: The output trajectories $y_i(t), i = 1, 2, 3, 4$ (dotted line represents $\tau = 0$, solid line represents $\tau = 2$) .

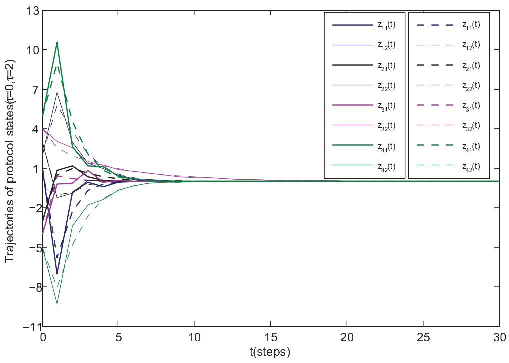

Fig. 8: Trajectories of protocol states $z_i(t), i = 1, 2, 3, 4$ (dotted line represents $\tau = 0$, solid line represents $\tau = 2$).

distributed protocol is proposed by using the networked predictive control method, the sufficient conditions of the output consensus are obtained, whose performance is verified by theoretical development as well as simulation examples. Future work can be done to obtain a condition not including A_i,B_i,C_i... of each agent and practical application of the proposed method.

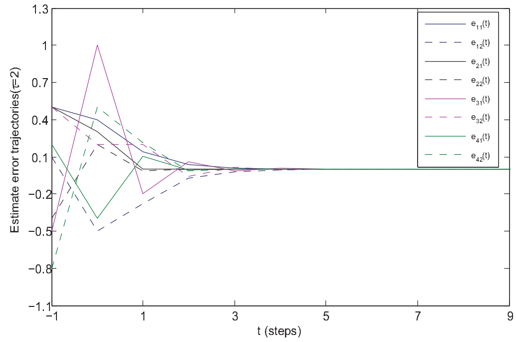

Fig. 9: Estimate error trajectories $e_i(t), i = 1, 2, 3, 4$ ($\tau = 2$).

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