Predictive Consensus for Networked Multi-agent Systems with Switching Topology and Variable Delay

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ABSTRACT

This paper investigates consensus problems for networked linear time invariant (LTI) multi-agent systems subjected to variable network communication delay and switching topology. A new protocol is proposed based on SIA matrix and predictive scheme for such systems with B that has full row rank. With predictive scheme the network delay is compromised. Consensus analysis based on *seminorm* is provided. Results give conditions for such systems with periodic switching topology to reach consensus. The proposed protocol allows for time-varying delay, switching topology, and unstable mode, numerical examples are given to demonstrate the effectiveness of the theoretical results.

Key Words: Networked multi-agent system, semi-norm, predictive scheme, switching topology, delay, consensus.

I. INTRODUCTION

Consensus problem has received extensive attention since last decade as it is deeply rooted in many fields like networked control and cooperative control of multi-agent systems. Consensus control, roughly speaking, is to devise protocols such that all the state of agents can be synchronized to a common state [1]. Since the proposition of the theoretic framework by Ofati-Saber and Murray [1], various consensus protocols have been proposed for different cases. For example, for multi-agent systems with first-order integrator dynamics and switching topologies, [2] derived necessary and sufficient condition for consensus-achieving, where there exists a spanning tree in the directed graph of the system. [3, 4, 5] studied the convergence of multi-agent system with double-integrator dynamics. Moreover, consensus protocols were also studied for multi-agent systems with linear time-invariant dynamics, where a spanning tree in the topology is usually required [8, 9].

The rapid development of network and communication technology brings the convenience of interaction and information exchange among different systems, and gives birth to networked multi-agent system (NMAS) and the Internet of things [11]. However, along with such convenience are the network constraints such as network delay and topology switching, which are the main difficulties of controlling multi-agent systems. In the last decade many works have been devoted to these difficulties. [12] studied asymptotic average consensus problem for multi-agent systems with time-varying delay, where each component is required to know the number of its out-neighbours. The consensus problem for Markovian jump second-order multi-agent systems with random communication delay and stochastic switching topology was discussed in [13]. [15] studied the average consensus problems of the discrete-time linear multi-agent systems with Markov switching topologies. Moreover, consensus analysis method in switching topology settings are mainly dependent on

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traditional approaches like Lyapunov mothod [16, 17], which is often hard to verify.

Different from the strategies to overcome network delay above, another appealing method is networked predictive scheme [18]. The main idea of this scheme is to use the observed states to predict the future states and compromise the network delay, and by doing so a equivalent system without delay is obtained, the prediction process is similar to that of Model predictive Control (MPC) aside from the optimization in MPC, which has been shown effective in synchronizing agents in environments free of delay [19, 20, 21, 22] including the integrator and LTI cases. The networked predictive scheme is proved effective in solving the consensus problem of multi-agent systems [23] in presence of constant network delay, and showed notable effectiveness for cases with variable delay [24]. However, to authors' knowledge, very little research work has been paid to the cases with topology switching of networked multi-agent systems so far. Moreover, in the study of consensus of multi-agent system with switching topology, a method worth mentioning is termed as *deterministic gossiping* [25], whose purpose is to enable agents to asymptotically determine, in a decentralized manner, the average of the initial values of their scalar gossip variables. In [25], *seminorm* analysis plays an important role in the analysis of consensus.

In this paper, based on predictive scheme and seminorm approach, a novel approach to syncrhonize the states of agents networked multi-agent system with switching topology and time-varying delay is proposed, contributions of this paper are the following: this paper expands the *seminorm* to vector gossip variables, providing a novel approach to perform direct consensus analysis for general linear states under topology switching conditions; based on seminorm approach, sufficient conditions and bounds for consensus are derived, which allow for asymmetric topology switching, time-varying delay and unstable mode. Numerical examples illustrate the effectiveness the proposed approach.

Paper organization: the first section gives literature review on networked predictive control and consensus with switching topology and delay. Some preliminaries of graph theory and *seminorm* are reviewed in the second section. Section III gives the main results. Numerical examples are presented in the next section. Section V concludes the paper.

II. PRELIMINARIES

To model a multi-agent system, a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is introduced with a node set $\mathcal{V}$ indexed by $i = 1, \cdots, n$, and an edge set $\mathcal{E}$ and a non-negative weighted matrix $\mathcal{A} = [a_{ij}]$. $a_{ji} > 0$ is meant that there exists communication from node i to node j and no communication otherwise. Node j is said to be a neighbour of node i if $a_{ij} > 0$. The Laplacian matrix $\mathcal{L}$ of graph $\mathcal{G}$ is defined by $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{A} = [a_{ij}]$ is the graph's weighted matrix, and $\mathcal{D}$ is diagonal matrix with its i-th diagonal element equal to $\sum_{j\in N_i} a_{ij}$. For notational convenience, $\beta(\mathcal{L})$ is meant the the corresponding graph of $\mathcal{L}$. A rooted graph is any graph with at least a node that has a directed path to each of the rest nodes[7, 26], a graph is strongly rooted if at least one node is adjacent to every node in the graph.

A non-negative square matrix $P = [p_{ij}]$ is said to be *stochastic* if all its row sums equal one. P is *scrambling* if it is stochastic and for each distinct pair of integer i and j, there exist is a column k of for which p_{ik} and p_{jk} are both non-zero. The ergodicity coefficient of P, denoted by $S(P)$, is defined by $S(P) = 1 - \min_{i,j} \sum_k \min\{p_{ik}, p_{jk}\}$ [6]. It is known that $S(P) < 1$ if and only if M is scrambling. Let $\|\cdot\|_p$ be an induced p-norm on $\mathbb{R}^{m\times n}$, the semi-norm of $M \in \mathbb{R}^{m\times n}$ is defined as $|M|_p = \min_{c\in\mathbb{R}^{1\times n}} \|M - \mathbf{1}c\|_p$. Seminorm shares triangle inequality [25] with norm, i.e.

$$|M_1 + M_2|_p \le |M_1|_p + |M_2|_p$$
$$|M_1 M_2|_p \le |M_1|_p |M_2|_p$$

In this paper semi-norm employs infinity norm unless otherwise noted, and its subscriber p is omitted for convenience*. For a vector is norm-bounded, it is also seminorm-bounded. For a stochastic matrix $A \in \mathbb{R}^{n\times n}$ and vectors x, y satisfying $x = Ay$, an inequality can be easily obtained that $|x| \le S(A)|y|$.

III. MAIN RESULTS

Consider a NMAS consisting of n agents indexed by $i \in I = \{1, \cdots, n\}$, where the dynamics of agent i is described by a discrete-time linear time-invariant system as follows:

$$\begin{aligned} x_i(t+1) &= Ax_i(t) + Bu_i(t), \\ y_i(t) &= Cx_i(t), i \in I, \end{aligned} \tag{1}$$

where $x_i(t) \in \mathbb{R}^m, u_i(t) \in \mathbb{R}^{m_u}, y_i(t) \in \mathbb{R}^{m_y}$ are the state, control input and the measured output of agent i, respectively, A, B, C are constant matrices with proper dimensions.

* For a scalar c, $|c|$ is still denoted its absolute value.

To model the communication of NMAS (1), a weighted digraph $\mathcal{G}(t) = \{\mathcal{V}(t), \mathcal{E}(t), \mathcal{A}(t)\}$ is introduced with node set $\mathcal{V}(t) = I$, adjacency matrix $\mathcal{A}(t) = [a_{ij}(t)]$. Similarly, the Laplacian matrix of Graph $\mathcal{G}(t)$, denoted by $\mathcal{L}(t)$, is defined by $\mathcal{L}(t) = \mathcal{D}(t) - \mathcal{A}(t)$, where and $\mathcal{D}(t)$ is the diagonal matrix with its i-th diagonal element equalling the $\sum_{j \in N_i} a_{ij}(t)$. In the paper, for the communication between agents, the following assumptions are imposed:

Assumption 1

A1: Agent i receives the output of agent i at each time t but with a communication delay $k_{ij}(t)$, which is variable and bounded by a constant τ.

A2: $k_{ij}(t)$ can be measured by a time-stamp, which requires that the clocks of all agents are synchronized.

A3: $a_{ij}(t)$ is bounded with $a_{ij}(t) \in \{0\} \cup [\underline{\alpha}, \overline{\alpha}]$, where $\underline{\alpha}, \overline{\alpha}$ are positive constants.

Denote $d_{ij}(t) = k_{ij}(t) - k_{ij}(t-1)$. It is usually assumed that the controllers always use the most recent data, thus, if at a sampling time t, the packet $P_{t-k_{ij}(t)}$ is used, then at time $t+1$ if there is no new arrival package, package $P_{t-k_{ij}(t)}$ will be still used. In other words, it always holds that $d_{ij}(t) \leq 1$, and further analysis gives that $d_{ij}(t) \in \{1, 0, -1 \cdots, -\tau\}$ [28].

Let $\hat{x}_{ij}(t - k_{ij}(t) + s|t - k_{ij}(t))$ be agent i's s step ahead prediction for agent j based on the information at time $t - k_{ij}(t)$. In particular, define the latest estimation $\bar{x}_{ij}(t - k_{ij}(t))$ to be $\hat{x}_{ij}(t - k_{ij}(t)|t - 1 - k_{ij}(t-1))$. Moreover, if $t - 1 - k_{ij}(t) = t - k_{ij}(t-1)$, let $\bar{x}_{ij}(t - k_{ij}(t)) = \bar{x}_{ij}(t - 1 - k_{ij}(t-1))$.

Since agent i receives information from agent j with a variable delay, prediction with constant delay will not be applicable in this case for two reasons: 1) dropouts, which happen when $d_{ij}(t) > 0$, causing the inconsistence of received data; 2) disorder of data arrival, which happens when data that are earlier sent is later received. To overcome the effect of variable network delay and make the prediction procedure work, based on the output data of agent j, the state predictions of agent j from time $t - k_{ij}(t)$ to t is made as follows:

$$\begin{aligned}\hat{x}_{ij}(t - k_{ij}(t) + 1|t - k_{ij}(t)) \\ = A\bar{x}_{ij}(t - k_{ij}(t)) + Bu_j(t - k_{ij}(t)) \\ + L(y_j(t - k_{ij}(t)) - C\bar{x}_{ij}(t - k_{ij}(t))),\end{aligned} \tag{2}$$

$$\begin{aligned}\hat{x}_{ij}(t - k_{ij}(t) + s + 1|t - k_{ij}(t)) \\ = A\hat{x}_{ij}(t - k_{ij}(t) + s|t - k_{ij}(t)) \\ + Bu_j(t - k_{ij}(t) + s),\end{aligned} \tag{3}$$

where $L \in \mathbb{R}^{m \times m_y}$ is the observer gain to be designed.

3.1. Error System

This subsection provides analysis for the error system of the proposed predictive scheme. First the following definitions are made for agent i

$$e_{ip}(j,t) \triangleq x_p(j+t) - \hat{x}_{ip}(j+t|t), \tag{4}$$

$$\begin{aligned}f_{ip}(t+1-k_{ip}(t+1)) \triangleq x_p(t+1-k_{ip}(t+1)) \\ - \bar{x}_{ip}(t+1-k_{ip}(t+1)) \\ = e_{ip}(1-d_{ip}(t+1), t-k_{ip}(t)),\end{aligned} \tag{5}$$

where $e_{ip}(j,t)$ is j-step ahead prediction error by agent i for agent p, $d_{ip}(t)$ is one-step ahead network delay difference with respect to agent p, $f_{ip}(t+1-k_{ip}(t+1))$ is the estimation error of agent p.

Replace t with $t - k_{ip}(t)$ in (1), and subtract it by (2), (3), it can be given that

$$\begin{aligned}e_{ip}(1, t-k_{ip}(t)) = (A - LC)f_{ip}(t - k_{ip}(t)), \\ e_{ip}(j+1, t-k_{ip}(t)) = Ae_{ip}(j, t-k_{ip}(t)), \\ j = 1, 2, \cdots, k_t.\end{aligned} \tag{6}$$

When $d_{ip}(t+1) = 1$, dropout happens, then

$$\begin{aligned}&f_{ip}(t+1-k_{ip}(t+1)) \\ &= x_{ip}(t+1-k_{ip}(t+1)) - \bar{x}_{ip}(t+1-k_{ip}(t+1)) \\ &= x_p(t-k_{ip}(t)) - \bar{x}_{ip}(t-k_t) = f_{ip}(t-k_{ip}(t)),\end{aligned}$$

otherwise, by multiplying the error, the estimation error becomes

$$\begin{aligned}&f_{ip}(t+1-k_{ip}(t+1)) \\ &= A^{-d_{ip}(t+1)}(A-LC)f_{ip}(t-k_{ip}(t))\end{aligned}$$

Thus the update of the error system for the proposed predictive scheme is given by

$$f_{ip}(t-k_t) = \Psi_{d_{ip}(t)} f_{ip}(t-1-k_{ip}(t-1)), \tag{7}$$

where

$$\Psi_m = \begin{cases} I & m = 1 \\ A^{-m}(A-LC) & otherwise \end{cases}.$$

Moreover, the error between the current state and the predicted state based on the latest estimation can be given by

$$\begin{aligned}x_p(t) - \hat{x}_{ip}(t|t-1-k_{ip}(t-1)) \\ = e_p(k_{ip}(t-1)+1, t-1-k_{ip}(t-1)) \\ = \Phi_{k_{ip}(t-1)} f_{ip}(t-1-k_{ip}(t-1)),\end{aligned} \tag{8}$$

where $\Phi_m = A^m(A-LC)$.

Consider the dynamics of $f_{ip}(t-k_t)$, the following result [28] is given

Theorem 1 *Suppose that Assumptions A1, A2, A3 hold true. Then, the error system $f_{ip}(t-k_{ip}(t))$ of variable-delay predictive scheme is asymptotically stable if there exists a positive definite matrix P such that*

$$\Psi_s^T P \Psi_s < P, s = -\tau, 1-\tau, \cdots, 0. \quad (9)$$

3.2. Coping with Topology Switching

This subsection deals with topology switching based on the predictive scheme.

Assumption 2 *Matrix B is with full row rank.*

Assumption 2 implies that B has its pseudo inverse. As an extension, this condition can replace by that $B(t)$ has full row rank. This assumption makes it convenient to preform consensus analysis based on results on consensus of integrator-type multi-agent systems, which will be clear later in the sequel. There are some practical examples showing that B can be of full row rank. Roughly speaking, fully-actuated like nonholonomic mobile robots [29], over-actuated linear systems belong to this type.

To cope with the topology switching, the consensus protocol is given by

$$\begin{aligned} u_i(t) &= \theta(B^T(BB^T)^{-1}A + K) \\ &\times \sum_{j=1}^{N} a_{ij}(t)(\hat{x}_{ij}(t|t-1-k_{ij}(t-1)) \\ &- \hat{x}_{ii}(t|t-1-k_{ii}(t-1))) \\ &+ K\hat{x}_{ii}(t|t-1-k_{ii}(t-1)), \end{aligned} \quad (10)$$

where $\theta \in \mathbb{R}^+$ and the gain matrix K are both to be designed.

When the topology is fixed, the protocol $u_i(t)$ can be directly devised [32]. However, when applying similar technique for the switching topology case, consensus achieving is hard to verify. Before embarking on consensus analysis, the following lemmas on seminorm are provided as follows.

Let p, q be positive integers. Consider q-dimensional vectors $a_i, b_i, c_i, i = 1, 2, \cdots, p$. Let M be a $p \times p$ stochastic matrix and F a $q \times q$ matrix. Let $|M|$ be the seminorm of M. Note that $|M|$ equals M's coefficient of ergodicity $S(M)$ [6]. Also note that for $v \in \mathbb{R}^n$, $|v| = \max_{i,j=1}^n (v_i - v_j)$, where v_j is the j-th element of v.

Let a_{ij}, b_{ij}, c_{ij} be the j-th element of a_i, b_i, c_i respectively. Define $p(a) = \sum_{i=1}^q |a^i|$, where

$$\begin{aligned} a^i &= [a_{1i} \quad a_{2i} \quad \cdots \quad a_{pi}]^T, \\ b^i &= [b_{1i} \quad b_{2i} \quad \cdots \quad b_{pi}]^T, \\ c^i &= [c_{1i} \quad c_{2i} \quad \cdots \quad c_{pi}]^T. \end{aligned}$$

The following lemma concerning $p(\cdot)$ is given

Lemma 1 *Let $c = (M \otimes F)a$, then there exits some positive integer d and a positive constant α_d such that $p(c) \le \alpha_d |M| \|F\|_d p(a)$, where $\|F\|_d$ denotes the d-norm of F.*

Proof 1 *First, let $c = (M \otimes I)b$. Since $c^i = Mb^i$, one has $|c^i| \le S(M)|b^i|$, which gives*

$$p(c) \le S(M)p(b).$$

Next let $b = (I_p \otimes F)a$, by the definition of seminorm

$$\begin{aligned} p(b) &= \sum_{i=1}^q |b^i| = \sum_{i=1}^q \max_{j,k=1}^p (b_{ji} - b_{ki}) \\ &= \sum_{i=1}^q \max_{j,k=1}^p (\sum_{l=1}^q f_{il}(a_{jl} - a_{kl})) \le \sum_{i=1}^q (\sum_{l=1}^q |f_{il}||a^l|) \\ &= \sum_{l=1}^q (|a^l| \sum_{i=1}^q (|f_{il}|)) \le \|F\|_1 p(a). \end{aligned}$$

Combining the two inequalities, the conclusion holds when $d = 1$, $\alpha_1 = 1$.

Remark 1 *There are other values that can be taken by d, $2, \infty$ for example, which is due to the equivalence between these norms [33]. In the sequel, 2-norm will be used with the corresponding $\alpha_2 = \sqrt{q}$. Moreover, from definition of $p(\cdot)$ it can be seen that if a is norm bounded, so is $p(a)$. Furthermore, if $p(a) = 0$, then a_is reach consensus. It is also easy that $p(a+b) \le p(a) + p(b)$.*

Remark 2 *Seminorm approach is originated from [25, 26], which is in scalar form. $p(\cdot)$ extends the scalar-form seminorm [25] to vector form for consensus analysis purpose, or purpose of analysis for dynamics of inter-agent state difference. Different from traditional consensus approaches like Lyapunov approach or matrix analysis method, seminorm approach is straightforward in deriving the connectivity conditions through the connectivity matrix by analyzing the inter-agent state difference, especially for the case with switching topology, when it is impossible to perform consensus analysis via traditional consensus analysis approaches.*

Lemma 2 *Consider a positive sequence $a(t)$ satisfying that $a(t) \le \mu\lambda^{t-t_0}a(t_0) + \mu\sum_{j=t_0}^{t}\lambda^{t-j}b(j)$, where $\mu > 0, 1 > \lambda > 0$ are positive constants, $b(j)$ is a bounded positive scalar sequence that convergences to zero, then $\lim_{t\to\infty} a(t) = 0$.*

Proof 2 *Since $\delta(t)$ is bounded, say $\sup_{t\geq 0}\delta(t)=M$,*

$$a(t)\leq \mu\lambda^t a(0)+\mu\sum_{j=0}^{t}\lambda^{t-j}b(j)\leq a(0)+\frac{\mu M}{1-\lambda}\triangleq \bar{M},$$

which implies $a(t)$ is bounded.

Now since $\delta(t)$ is a diminishing scalar sequence, there exists a t_ϵ such that for all $t>t_\epsilon$, $\delta(t)\leq\epsilon$,

$$\begin{aligned}a(t)&\leq \mu\lambda^{t-t_\epsilon}a(t_\epsilon)+\mu\sum_{j=t_\epsilon}^{t}\lambda^{t-j}b(j)\\&\leq \mu\lambda^{t-t_\epsilon}\bar{M}+\frac{\mu\epsilon}{1-\lambda}.\end{aligned}$$

Let t goes to infinity, by the arbitrariness of ϵ, it can be seen that $\lim_{t\to\infty}a(t)=0$.

Combine (10) and (1), one obtains

$$\begin{aligned}x_i(t+1)&=Ax_i(t)+BK\hat{x}_{ii}(t|t-1-k_{ii}(t-1))\\&+\theta(A+BK)\sum_{j=1}^{N}a_{ij}(t)(\hat{x}_{ij}(t|t-1-k_{ij}(t-1))\\&-\hat{x}_{ii}(t|t-1-k_{ii}(t-1))).\end{aligned}$$

To preceed, let $l_i(t)$ be the i-th row of $\mathcal{L}(t)$ and m_j the j-th row of I_n. Considering the error system of predictive scheme given in (7), one can derive the compact form of the closed system

$$\begin{aligned}x(t+1)&=(\mathcal{H}(t)\otimes(A+BK))x(t)\\&\quad+\Omega(t)\Phi(t)f(t-1),\\f(t)&=\Psi(t)f(t-1),\end{aligned}\tag{11}$$

where

$$\begin{aligned}\mathcal{H}(t)&=I_n-\theta\mathcal{L}(t),\\x(t)&=\begin{bmatrix}x_1^T(t) & x_2^T(t) & \cdots & x_n^T(t)\end{bmatrix}^T,\\f(t)&=\begin{bmatrix}f_1^T(t) & f_2^T(t) & \cdots & f_n^T(t)\end{bmatrix}^T,\\f_i(t)&=\begin{bmatrix}f_{i1}(t-k_{i1}(t))\\f_{i2}(t-k_{i2}(t))\\\vdots\\f_{in}(t-k_{in}(t))\end{bmatrix},\\\Psi(t)&=\mathrm{diag}\{\Psi_1(t),\Psi_2(t),\cdots,\Psi_n(t)\},\\\Psi_i(t)&=\mathrm{diag}\{\Psi_{k_{i1}(t)},\Psi_{k_{i2}(t)},\cdots,\Psi_{k_{in}(t)}\},\\\Phi(t)&=\mathrm{diag}\{\Phi_1(t),\Phi_2(t),\cdots,\Phi_n(t)\},\\\Phi_i(t)&=\mathrm{diag}\{\Phi_{k_{i1}(t)},\Phi_{k_{i2}(t)},\cdots,\Phi_{k_{in}(t)}\},\\\bar{\mathcal{L}}(t)&=\mathrm{diag}\{l_1(t),l_2(t),\cdots,l_n(t)\},\\\bar{\mathcal{I}}_n&=\mathrm{diag}\{m_1,m_2,\cdots,m_n\},\\\Omega(t)&=-\theta\bar{\mathcal{L}}_n(t)\otimes(A+BK)-\bar{\mathcal{I}}_n\otimes(BK).\end{aligned}$$

Obviously $\Omega(t)$ is norm-bounded. Moreover, it can be seen that $\mathcal{H}(t)$ is stochastic, and a quick thought is to employ the properties of SIA matrix.

Let $\delta(t)=\Omega(t)\Phi(t)f(t-1),\Phi(t,\tau)=\prod_{i=\tau}^{t}(\mathcal{H}(i)\otimes(A+BK))$. † If there is $y=\Phi(t,\tau)x$, it is easy to show that

$$p(y)\leq\sqrt{n}\|A+BK\|_2^{t-\tau}S(\prod_{j=\tau}^{t}\mathcal{H}(j))p(x).\tag{12}$$

Moreover, one has

$$x(t)=\Phi(t,0)x(0)+\sum_{j=0}^{t-1}\Phi(t,t-j-1)\delta(j).\tag{13}$$

It is assumed that condition (9) holds. In fact so far it is enough to present conditions of $\mathcal{H}(t)$ and $A+BK$ for the multi-agent system to reach consensus by applying $p(\cdot)$ on both sides of (13). Instead, to further investigate the joint connectivity of the graph and get a more general result for the case with switching topology, the following transformation is first presented. By iteration, one has

$$x(t+d)=\Phi(t+d,t)x(t)+\tilde{\delta}(t),\tag{14}$$

where $\tilde{\delta}(t)=\sum_{j=t}^{t+d-1}\Phi(t+d,j)\delta(j)$, and d is an given constant integer.

Let (9) hold, which implies that $\delta(t)$ is a diminishing signal, one also has that there exists some positive $\underline{c},\bar{c}$ such that

$$\underline{c}\|\delta(t)\|\leq\|\tilde{\delta}(t)\|\leq\bar{c}\|\delta(t)\|,\tag{15}$$

where $\bar{c}=\bar{c}(\|A+BK\|),\underline{c}=\underline{c}(\|A+BK\|)$.

Now it can be seen that $\|\tilde{\delta}(t)\|,r=0,1,2,\cdots$ is also a bounded sequence that converges to zero since $\|A+BK\|$ is bounded. Moreover, $p(\tilde{\delta}(t))$ is bounded and converges to zero.

To preceed, define $s_d(t)=S(\prod_{j=t}^{t+d-1}\mathcal{H}(j))$. Note that $s_d(t)$ means the overall connectivity over the d steps in the future, for which it is called the connectivity index w.r.t. d in the sequel.

Some basics on joint connectivity are presented here. Consider an infinite strictly increasing sequence of integer $t_r,r=0,1,\cdots$ with $t_0=0$ and $t_{r+1}-t_r\leq d$, where d is a positive integer. Suppose that for each pair t_r and t_{r+1}, there exists a strictly increasing sequence of integer $t_{r,j},j=0,1,\cdots,m_r$ with $t_{r,0}=t_r$ and $t_{r,m_r}=r_{r+1}$ satisfying $t_{r,j+1}-t_{r,j}\geq T_2$ for

† Here the product is meant the left multiplication of matrix.

some integer $m_r \geq 0$. Suppose that the communication graph $\mathcal{G}_\sigma$ remains unchanged during each interval $[t_{r,j}, t_{r,j+1})$. Obviously there are at most $m_\star = \lfloor \frac{T_1}{T_2} \rfloor$ subintervals at each interval $[t_r, t_{r+1})$, where $\lfloor \frac{T_1}{T_2} \rfloor$ denotes the maximum integer no larger than $\frac{T_1}{T_2}$. Denote $\mathcal{G}_{t_r} = \mathcal{G}_{r,m_r-1} \circ \mathcal{G}_{r,m_r-2} \circ \cdots \circ \mathcal{G}_{r,0}$. $\mathcal{G}_{t_r}$ is said to be repeatedly jointly (strongly) rooted with d if $\mathcal{G}_{t_r}$ is (strongly) rooted for all $r \geq 0$.

For simplicity, let $t_r = rd, r \in \mathbb{N}$. Impose another assumption on the communication graph

Assumption 3 *$\mathcal{G}_{t_r}$ is repeatedly jointly rooted.*

Such connectivity will be called long-run connectivity w.r.t. d in the sequel. Long-run connectivity can be consistent with predictive scheme, which requires at least one packet is received in each interval for each edge of the joint graph, for details see the connectivity in [27]. Moreover, define $s_d = \sup_{r \geq 0} s_d(rd)$ to be the *long-run connectivity index w.r.t. d*. Obviously $s_d \leq 1$. For network that does not possess such connectivity, we say its long-run connectivity w.r.t. d is 1.

Now we are ready to state our main result.

Theorem 2 *Consider a NMAS described by (1). Let assumption 1, 2 and 3 hold. Let s_d be the long-run connectivity index of this system w.r.t. d. The protocol (10) solves the consensus problem of this multi-agent system if the following conditions are satisfied.*

- *C1 Condition (9) holds.*
- *C2 There exists some K with proper dimensions such that*

$$\begin{bmatrix} -s_d^{-\frac{1}{(n-1)d}} I & A + BK \\ A^T + K^T B^T & -s_d^{-\frac{1}{(n-1)d}} I \end{bmatrix} < 0. \tag{16}$$

Proof 3 *Let $\tilde{s}_r = \prod_{j=dr}^{dr+d-1} \mathcal{H}(j)$. Since $\theta \leq 1/\bar{d}$, one has that $\mathcal{H}(t)$ is a stochastic matrix with positive diagonal elements.*

Let $\mathcal{G}(H)$ be the graph in the sense that there is a directed edge from j to i if $h_{ij} > 0$, where $H = [h_{ij}]$ is nonnegative square matrix. Since Assumption 3 holds, by Lemma 2 of [30] one has $\tilde{s}_r$ is with all diagonals positive and consequently $\mathcal{G}(\tilde{s}_r)$ is rooted and self-arced. Next by Lemma 6.28 of [31] it can be obtained that $\mathcal{G}(\tilde{s}_{r+n-2}) \circ \mathcal{G}(\tilde{s}_{r+n-3}) \circ \cdots \mathcal{G}(\tilde{s}_r)$ is scrambling. Let $\tilde{u}_r = \prod_{j=r}^{r+n-2} \tilde{s}_j$, then applying Lemma 3.4 of [34], one has

$$S(\tilde{u}_j) \leq 1 - s_d^{(n-1)d}, \forall\, j \tag{17}$$

By iteration, one has

$$\begin{aligned} x(t_a + (n-1)d) &= \Phi(t_a + (n-1)d, t_a)x(t_a) + \eta(t_a) \\ &= (\tilde{u}_a \otimes (A + BK)^{(n-1)d})x(t_a) + \eta(t_a), \end{aligned}$$

where $\eta(t_a) = \sum_{j=t_a}^{t_a+(n-1)d-1} \Phi(t_a + (n-1)d, j)\delta(j)$.

Let $a = b + k(n-1)d, \tilde{v}_a = \tilde{u}_a \otimes (A + BK)^{(n-1)d}$, by (14),

$$\begin{aligned} x(t_a) = &(\prod_{j=0}^{k-1} \tilde{v}_{r+j(n-1)d})x(t_b) \\ &+ \sum_{j=0}^{k-1}((\prod_{l=j}^{k-1} \tilde{v}_l)\eta(t_{a+l(n-1)d})). \end{aligned} \tag{18}$$

Apply $p(\cdot)$ on both sides of (18), by (17), it can be obtained that

$$\begin{aligned} p(x(t_a)) \leq &\mu\lambda^k p(x(t_b)) \\ &+ \mu \sum_{j=0}^{k-1} \lambda^{k-1-j} p(\tilde{\delta}(t_{a+j(n-1)d})), \end{aligned} \tag{19}$$

where $\mu = \sqrt{m}, \lambda = s_d^{(n-1)d}\|A + BK\|_2^{(n-1)d}$.

Let $C2$ hold, it can be given that λ defined in (19) is less that one. Let C1 hold, one has $\eta(t_j)$ in (19) is a diminishing signal that converges to zero. Fix b, applying Lemma 2 gives that $p(x(rd))$ converges to zero as r goes to infinity. Considering that there exists some positive constants $\bar{k}_1, \bar{k}_2$ such that $p(x(rd+i)) \leq \bar{k}_1 p(x(rd)) + \bar{k}_2 p(\tilde{\delta}(rd)), i = 1, 2, \cdots, d-1$, it follows that $\lim_{t\to\infty} p(x(t)) = 0$, completing the proof.

Remark 3 *Although $C2$ can also be given in equivalent form of 2-norm, F-norm for example, which for content it omitted ; when network delay is constant, i.e., $d(t) = 0$, $C1$ becomes that $A - LC$ is Schur; obviously B is cancelled in the above proof because B has full row rank, making it possible to carry out seminorm analysis and apply results on consensus of integrator multi-agent systems; by solving (16), the gain matrix K can be obtained, which, however, requires that each agent knows the bounds of $a_{ij}(t)$, which in some sense breaks the philosophy of distributed control design.*

3.3. System Stability

This subsection gives a brief discussion on the system stability under the proposed consensus protocol. The main result is, under the proposed protocol, consensus means stability if the network is not jointly rooted, however, if the network has some periodic joint connectivity, consensus and stability may not be obtained at the same time.

First, when the network is not jointly rooted, the consensus condition becomes that $A + BK$ is Schur. On the other hand, since $\sigma_{max}(\mathcal{H}(t))$ is no large than one, one can easily choose a $P > 0$ with proper dimensions such that $(\mathcal{H}(t) \otimes (A + BK))^T P(\mathcal{H}(t) \otimes (A + BK)) - P < 0$, thus stability can also be gained. However, when some connectivity exists in the graph, say the condition that the graph is jointly strongly rooted with a constant $d > 0$, it is easy to show that $s_d < 1$, when there is no guarantee that $A + BK$ is Schur. For the stability of the multi-agent system in this case, the readers are suggested to refer to [35] and references therein.

Graph connectivity expands the synchronizing region for the choice of K [36], which may also bring instability the system, as can be seen in the numerical examples in the following.

IV. NUMERICAL EXAMPLES

Consider a multi-agent system described by (1) consisting of $n = 4$ agents indexed by $1, 2, 3, 4$, the states of each agent represent the first two heights of a quadruple tank [37]. The initial states of the agents are randomly generated and the dynamics of agent $i, i = 1, 2, 3, 4$ is given as follows

$$x_i(t+1) = \begin{bmatrix} 0.9815 & 0.0184 \\ 0.0184 & 0.9711 \end{bmatrix} x_i(t) + \begin{bmatrix} 225.2 & 0.7031 \\ 2.109 & 74.65 \end{bmatrix} u_i(t)$$

$$y_i(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_i(t)$$

Using the predictive scheme to design observer, the gain L is given by $L = \begin{bmatrix} -0.0227 & 0.4959 \\ 0.3374 & -0.7358 \end{bmatrix}$.

4.1. Example 1

Choose $\theta = 0.2$, calculation shows that $s_d = 0.9507$. Use the proposed protocol to synchronize the states of agents, the gain K is given by $\begin{bmatrix} -0.0009 & -0.0005 \\ -0.0016 & -0.0019 \end{bmatrix}$, the eigenvalues are 0.6921 and 0.9039, which lie within the unit ball. The simultion shows that consensus can be reached, as is shown by fig. 4, and also shows that the system is stable, as shown by fig. 5.

Let $a_{ij}(t) \in \{0\} \cup [1, 2]$. The topology switches between the four toplogies in Fig. 1, satisfying that for each interval $6s \leq t < 6s + 6, s \in \mathbf{N}$, each of the four topologies in Fig. 1 appears at least once.

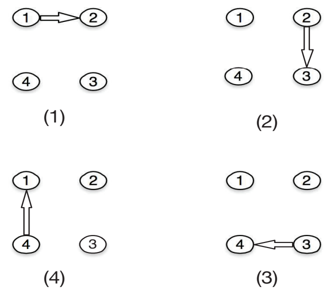

Fig. 1. Topologies

It is obvious that the topology of the system, though switches, is periodically jointly rooted. Subject system to random delay with a upper bound $\tau = 2$, i.e., each transmission of between agents is subjected a random network delay no larger than 2.

It is also obvious that this kind of connectivity, compared to those in [8, 12, 15], is asymmetrically switching with time-varying delay. The simulation shows that consensus can be reached, as is shown by fig. 2, and also shows that the system is stable, as is shown in fig. 3.

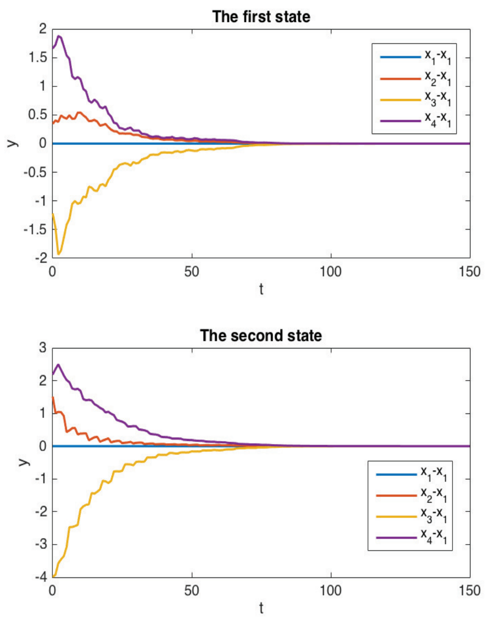

Fig. 2. Consensus error

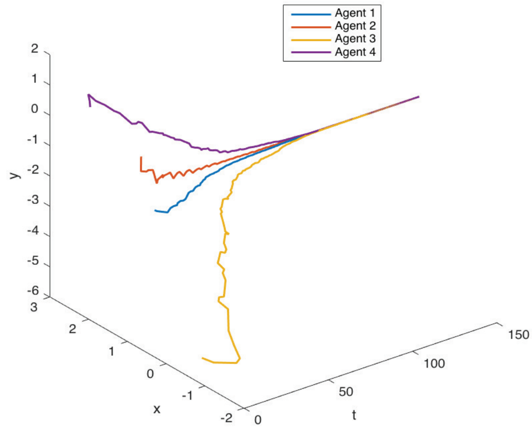

Fig. 3. State trajectory

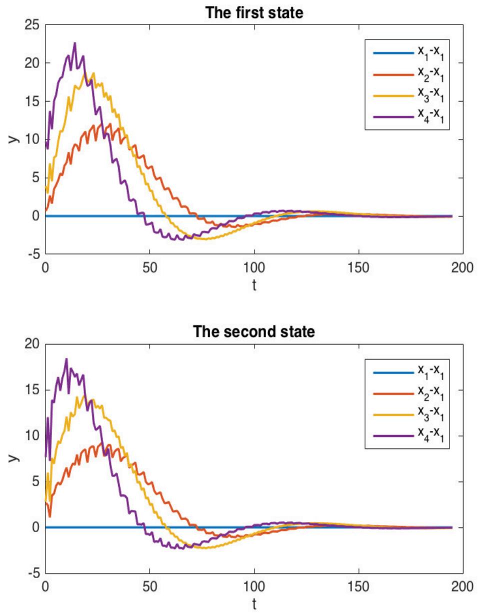

Fig. 4. Consensus error

4.2. Example 2

As another comparision, this example still uses the same parameters in the last example except the gain matrix K, which is given by $\begin{bmatrix} -0.0025 & 0.0035 \\ 0.0104 & -0.0135 \end{bmatrix}$, the eigenvalues of $A + BK$ are 1.022 and -0.6239, which means the system contains unstable mode. Still the simultion shows that consensus can be reached, the eigenvalues of $A + BK$ are 1.022 and -0.6239, as is shown by fig. 4, yet the system is unstale, as shown by fig. 5. At last, in fact, unlike [16, 8], consensus can be directly checked by tunning the eigenvalues of $A + BK$, which is also an advantage.

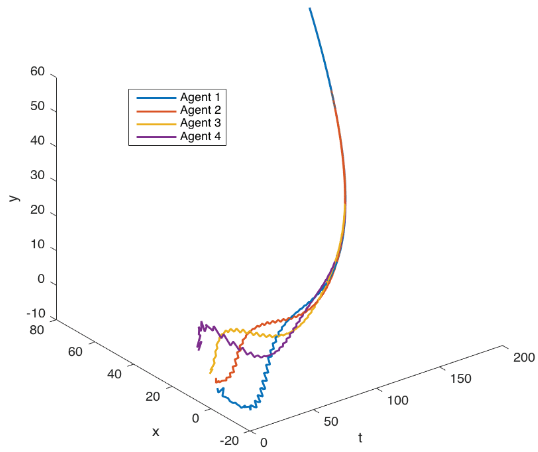

Fig. 5. State trajectory

V. CONLUSION

Consensus is a key problem in control of NMAS and is difficult to solve because of the delay and topology switching within. For NMAS with variable delay, predictive scheme proves a successful tool to overcome delay; seminorm provides an effective method for consensus analysis. Together with them both, this problem can be solved in a nice way. Our next step will be focused on more complicated cases like when the agent dynamics are non-identical, and relaxation of B will also be considered.

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