# Joint state and fault estimation for time-varying nonlinear systems with randomly occurring faults and sensor saturations ⋆

Jun Hu [a,b], Zidong Wang [c], Huijun Gao [d]

[a] *Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, China*

[b] *School of Engineering, University of South Wales, Pontypridd CF37 1DL, UK*

[c] *Department of Computer Science, Brunel University London, Uxbridge, Middlesex, UB8 3PH, UK*

[d] *Research Institute of Intelligent Control and Systems, Harbin Institute of Technology, Harbin 150001, China*

**Abstract**

This paper is concerned with the joint state and fault estimation problem for a class of uncertain time-varying nonlinear stochastic systems with randomly occurring faults and sensor saturations. A random variable obeying the Bernoulli distribution is used to characterize the phenomenon of the randomly occurring faults and the signum function is employed to describe the sensor saturation due to physical limits on the measurement output. The aim of this paper is to design a locally optimal time-varying estimator to simultaneously estimate both the system states and the fault signals such that, at each sampling instant, the covariance of the estimation error has an upper bound that is subsequently minimized by properly designing the estimator gain. The explicit form of the estimator gain is characterized in terms of the solutions to two difference equations. It is shown that the developed estimation algorithm is of a recursive form that is suitable for online computations. In addition, the performance analysis of the proposed estimation algorithm is conducted and a sufficient condition is given to verify the exponential boundedness of the estimation error in the mean square sense. Finally, an illustrative example is provided to show the usefulness of the developed estimation scheme.

*Key words:* Time-varying nonlinear systems; Fault estimation; Randomly occurring faults; Sensor saturations; Recursive matrix difference equations.

## 1 Introduction

In modern large-scale industrial systems, the system states are not always available and the measurement outputs are often subject to stochastic noises due mainly to physical constraints, costs for measuring and environmental complexities [5, 12]. Therefore, the state estimation or filtering problem has long been one of the foundational research problems in signal processing and control areas that has received a great deal of research attention [16, 29]. The past decades have witnessed the rapid development of various estimation and filtering algorithms that have been successfully applied in engineering practice such as signal processing, navigation and control of vehicles, guidance and econometrics. According to the performance indices, the filtering algorithms can be generally categorized into the main streams of Kalman filtering [4, 18], extended Kalman filtering [26,30], particle filtering [32], set-valued filtering [6], $H_\infty$ filtering [31, 37], and non-Gaussian filtering [9]. To be more specific, the celebrated Kalman filtering algorithm has been proposed in [18] for linear stochastic systems with Gaussian noises. Based on polynomial observations, the mean-square filters have been designed in [1,2] for nonlinear polynomial systems with, respectively, white Poisson processes and Wiener processes. In [30], the extended Kalman filtering approach has been developed for nonlinear dynamic gene regulatory networks to identify the model parameters and the actual value of gene expression levels. The stochastic stability of the extended Kalman filter has been discussed in [26] for nonlinear stochastic systems, and the corresponding results have then been extended to the case where the measurement outputs suffer from intermittent observations. Recently, the $H_\infty$ filter has been constructed in [37] for a class of discrete-time linear switched systems with the persistent dwell-time switching signals.

On another research frontier, sensors may not always be capable of providing signals with unlimited amplitudes due to physical/technological restrictions. The occurrence of the sensor saturations could impose severe degradations on the system performance if not handled properly [7, 31]. Consequently, the filtering problems with sensor saturations have been gaining some initial research attention and some preliminary results have appeared in recent literature [25, 31]. The main challenge with this topic is how to design a fil-

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⋆⋆Corresponding author: Jun Hu.

*Email addresses:* hujun2013@gmail.com (Jun Hu), Zidong.Wang@brunel.ac.uk (Zidong Wang).

tering algorithm by making full use of the available information about the sensor saturations (e.g. types, intensities and distributions) subject to specified performance requirements (e.g. minimized variance and guaranteed $H_\infty$ constraints). For example, in [7], the fault detection filter has been designed for discrete-time Markovian jump systems with sensor saturations, incomplete knowledge of transition probabilities and randomly varying nonlinearities. The state estimation problem has been considered in [14] for a class of time-invariant systems with distributed sensor delays by using a linear matrix inequality (LMI) approach, where the performance specification (i.e. variance) and sensor saturations have not been considered. Recently, an effective $H_\infty$ filtering algorithm has been developed in [31] to address the phenomena of the randomly occurring sensor saturations and missing measurements in a unified framework. It is worth mentioning that, so far, most reported results have been concerned with *time-invariant systems* only and the corresponding filter design issue for *time-varying* systems with variance constraints has not been paid adequate research attention despite the fact that almost all real-world systems have certain structures/parameters that are time-varying.

Apart from the sensor saturations, component faults constitute another common cause for performance deteriorations or even instability of the engineering systems [12, 13, 28, 33, 34]. Therefore, in the past decade, considerable research effort has been devoted to the fault detection and estimation (FDE) problems, see e.g. [17, 19, 20, 35] and the references therein. Among others, a new estimation algorithm based on augmented approach has been presented in [12] for descriptor nonlinear systems with sensor fault and efficient fault-tolerant control approach with compensation mechanism has been developed in [39] for singular systems with actuator saturation and nonlinear perturbation. The sensor fault-tolerant speed tracking control scheme has been given in [23] for an electric vehicle powered by a permanent-magnet synchronous motor and novel estimation method has been proposed in [36] for Takagi-Sugeno fuzzy model with time-varying sensor fault. Moreover, in [19, 20], the fault detect filters have been designed for uncertain systems with mixed time-varying delays and nonlinear perturbations by using LMI method. Recently, the FDE problems for *time-varying* systems have stirred much research interest owing to the increasing importance of the time-varying behaviors in practical system modeling. Up to now, a few efficient FDE schemes have been proposed for linear/nonlinear time-varying systems. For example, the finite-horizon $H_\infty$ fault estimation problems have been studied in [27, 38] for linear discrete time-varying stochastic systems by using the Krein-space theory.

In parallel to the recent development of the networked control systems [11], some initiatives have been made on the network-induced nature of the fault signals. Recently, it has been shown in [8] that the occurrence of the faults could be intermittent or even random especially in networked environments due to unpredictable parameter fluctuations or structural changes over the networks. In [8], the effects from the randomly occurring faults (ROFs) onto the estimation performance have been examined by proposing an $H_\infty$ fault estimation algorithm over a finite horizon, where a backward recursive Riccati difference equation approach has been employed. So far, to the best of the authors' knowledge, the problem of *joint* state and fault estimation problem for *time-varying* systems with *ROFs* has not been addressed yet, not to mention the case when the underlying system is also subject to sensor saturations, parameter uncertainties as well as nonlinearities. Besides, it should be mentioned that most of existing methods fail to provide the performance analysis of estimation algorithm for addressed time-varying nonlinear systems with certain complexities. As such, the purpose of this paper is to shorten such a gap by developing a design scheme for effective estimators capable of jointly estimating system states and fault signals with help from the difference equation method and presenting a performance analysis criterion.

Motivated by the above discussions, in this paper, we aim to investigate the problem of joint state and fault estimation for a class of uncertain time-varying nonlinear systems with ROFs and sensor saturations. The phenomenon of randomly occurring fault is characterized by using a Bernoulli random variable with known occurrence probability. The focus is on designing a recursive estimator to simultaneously estimate the system states and fault signals such that, for all admissible parameter uncertainties, nonlinearities, ROFs and sensor saturations, an upper bound of the estimation error covariance is guaranteed and then minimized at each time step by properly choosing the estimator gain. The main novelties of this paper are highlighted as follows: 1) the addressed model is comprehensive which accounts for several well-known phenomena (i.e. parameter uncertainties, nonlinearities, ROFs and sensor saturations) contributing to system complexities in a unified framework; 2) a new compensation scheme is proposed to attenuate the effects from both the ROFs and the sensor saturations onto the estimation performance; 3) a sufficient criterion is given to quantify the boundedness analysis of the estimation error in the mean square sense and an upper bound of the bias of developed estimation is presented; and 4) the designed estimation algorithm is of a recursive feature suitable for online computations. Finally, simulations are provided to demonstrate the usefulness of proposed estimation method.

*Notations*. The notations used throughout this paper are standard. $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. $\|\cdot\|$ is the Euclidian norm of real vectors or the spectral norm of real matrices. For a matrix $P$, $P^T$ and $P^{-1}$ represent its transpose and inverse, respectively. $\mathbb{E}\{x\}$ is the mathematical expectation of the stochastic variable $x$ and $\mathbb{E}\{x|y\}$ is the mathematical expectation of the stochastic variable $x$ conditional on $y$. $I$ and 0 stand for the identity matrix and the zero matrix with appropriate dimensions, respectively. $\mathrm{tr}(P)$ represents the trace of matrix $P$. $\mathrm{diag}\{P_1, P_2, \cdots, P_N\}$ represents a block-diagonal matrix with matrices $P_1, P_2, \cdots, P_N$ on the diagonal. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

## 2 Problem Formulation and Preliminaries

In this paper, we consider the following class of discrete uncertain time-varying nonlinear systems:

$$\vec{x}_{k+1} = (\vec{A}_k + \Delta\vec{A}_k)\vec{x}_k + \vec{g}(\vec{x}_k) + \alpha_k\vec{F}_k f_k + \vec{B}_k\omega_k \qquad (1)$$

$$\vec{y}_k = \sigma(\vec{C}_k\vec{x}_k) + \nu_k \tag{2}$$

where $\vec{x}_k \in \mathbb{R}^n$ represents the system state, $\vec{x}_0$ is the initial value with mean $\bar{\vec{x}}_0$, $\vec{y}_k \in \mathbb{R}^m$ is the measurement output, $f_k \in \mathbb{R}^f$ is the fault signal, $\omega_k$ is the zero-mean process noise with covariance $Q_k > 0$, and $\nu_k$ is the zero-mean measurement noise with covariance $R_k > 0$. $\vec{A}_k$, $\vec{F}_k$, $\vec{B}_k$ and $\vec{C}_k$ are known and bounded matrices with appropriate dimensions. $\Delta\vec{A}_k$ is a real-valued uncertain matrix satisfying

$$\Delta\vec{A}_k = \vec{M}_k\mathcal{F}_k\vec{N}_k, \;\; \mathcal{F}_k\mathcal{F}_k^T \le I \tag{3}$$

where $\mathcal{F}_k$ represents the time-varying uncertainty, $\vec{M}_k$ and $\vec{N}_k$ are known time-varying matrices with appropriate dimensions. The known nonlinear function $\vec{g}(\vec{x}_k)$ satisfies

$$\|\vec{g}(u) - \vec{g}(v)\| \le \|G(u-v)\|, \;\; \forall\, u, v \in \mathbb{R}^n \tag{4}$$

where $G$ is a known matrix.

It is assumed that the dynamic characteristics of the fault $f_k$ are represented by

$$f_{k+1} = A_{f,k} f_k \tag{5}$$

where $A_{f,k}$ is a known matrix with appropriate dimension. It is easy to see that the fault becomes a constant one if $A_{f,k} \equiv I$. The random variable $\alpha_k \in \mathbb{R}$, which characterizes the phenomenon of the randomly occurring fault, satisfies the Bernoulli distribution taking the values of 0 or 1 with

$$\begin{aligned} &\text{Prob}\,\{\alpha_k = 1\} = \mathbb{E}\,\{\alpha_k\} = \bar{\alpha}, \\ &\text{Prob}\,\{\alpha_k = 0\} = 1 - \bar{\alpha}, \end{aligned} \tag{6}$$

where $\bar{\alpha} \in [0, 1]$ is a known scalar. Throughout this paper, we assume that $\alpha_k$, $\omega_k$, $\nu_k$ and $\vec{x}_0$ are mutually independent.

The saturation function $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is defined as:

$$\sigma(v) = \begin{bmatrix} \sigma_1(v_1) & \sigma_2(v_2) & \cdots & \sigma_m(v_m) \end{bmatrix}^T, \tag{7}$$

where $\sigma_i(v_i) = \text{sign}(v_i)\min\{\varrho_i, |v_i|\}$, $\text{sign}(\cdot)$ denotes the signum function and $\varrho_i$ $(i = 1, 2, \cdots, m)$ is the saturation level.

**Remark 1** *The assumption (4) has been widely investigated in a large body of literature. It is worthwhile to notice that, under the same assumption, the problems of fault detection and fault estimation have been extensively studied for complex dynamical systems subject to the Lipschitz nonlinearities, see e.g. [3, 22].*

**Remark 2** *In this paper, the fault estimation problem is studied for a class of time-varying nonlinear systems, where the additive fault is addressed. It is worthwhile to note that the fault model in (5) depicts that the faults can dynamically change with hope to better reflect the engineering reality. In fact, the faults in (5) could include the constant faults as a special case [8]. Besides, we use a linear fault dynamics with time-varying $A_{f,k}$ to characterize the time-varying nature of the system addressed, which can be applicable to the case after the fault detection. Here, we assume that $A_{f,k}$ is a known matrix. The reason is that we can obtain certain priori knowledge of the fault based on the engineering background after the stage of the fault detection, see e.g. the incipient faults and constant faults. Such kind of fault is fairly common in real world, see e.g. [21]. On the other hand, we discuss the phenomenon of the randomly occurring fault in (1) by employing a Bernoulli distributed random variable $\alpha_k$. That is, the fault occurs if $\alpha_k = 1$, and there is no fault if $\alpha_k = 0$. It is worthwhile to note that the randomness of the fault lies in its effects onto the addressed systems rather than the "amplitude" of the fault signal.*

**Remark 3** *The aim of the addressed problem is to develop a fault estimation scheme after the fault is detected. For example, when the sudden fault at random instants occurs, the sudden change (i.e. the fault signal) would be firstly detected by utilizing the existing fault detection algorithm, and then the fault estimation scheme proposed in this paper would be implemented. In this case, the initial execution time of our developed estimation algorithm is equivalent to the sudden change time. Summarizing the above discussions, the sudden change could be regarded as the emergence of the fault, which could be reflected by fault detection schemes. The fault model (5) describes the dynamics of the fault signal, which could characterize the change of the "amplitude" of the fault signal. The main idea of our developed estimation scheme is to generate the fault estimation after the fault is detected (e.g. after the sudden change time). On the other hand, it should be mentioned that it is vital important to discuss the uncertain fault model when the priori knowledge of the fault is inaccurate. Hence, we will consider the possibility of extending our results to address the uncertain fault model, which constitutes one of the future research directions.*

Setting $x_k = \begin{bmatrix} \vec{x}_k^T & f_k^T \end{bmatrix}^T$, we have

$$\begin{aligned} x_{k+1} = {} & [\bar{A}_k + \bar{A}_{1,k} + (\alpha_k - \bar{\alpha})\bar{A}_{2,k}]x_k + g(x_k) \\ & + B_k\omega_k \end{aligned} \tag{8}$$

$$y_k = \sigma(C_k x_k) + \nu_k \tag{9}$$

where

$$\begin{aligned} &\bar{A}_k = \begin{bmatrix} \vec{A}_k & \bar{\alpha}\vec{F}_k \\ 0 & A_{f,k} \end{bmatrix}, \;\; \bar{A}_{1,k} = \begin{bmatrix} \Delta\vec{A}_k & 0 \\ 0 & 0 \end{bmatrix}, \\ &\bar{A}_{2,k} = \begin{bmatrix} 0 & \vec{F}_k \\ 0 & 0 \end{bmatrix}, \;\; g(x_k) = \begin{bmatrix} \vec{g}(\vec{x}_k) \\ 0 \end{bmatrix}, \\ &B_k = \begin{bmatrix} \vec{B}_k \\ 0 \end{bmatrix}, \;\; C_k = \begin{bmatrix} \vec{C}_k & 0 \end{bmatrix}, \;\; y_k = \vec{y}_k. \end{aligned} \tag{10}$$

In this paper, we construct the following time-varying estimator:

$$\hat{x}_{k+1|k} = \bar{A}_k\hat{x}_{k|k} + g(\hat{x}_{k|k}) \tag{11}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - C_{k+1}\hat{x}_{k+1|k}) \tag{12}$$

where $\hat{x}_{k|k}$ is the estimation of $x_k$ at time $k$ with $\hat{x}_{0|0} = \begin{bmatrix} \bar{\bar{x}}_0^T & 0 \end{bmatrix}^T$, $\hat{x}_{k+1|k}$ is the one-step prediction at time $k$, and $K_{k+1}$ is the estimator gain to be determined.

Let $\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1}$ be the estimation error and $P_{k+1|k+1} = \mathbb{E}\{\tilde{x}_{k+1|k+1}\tilde{x}_{k+1|k+1}^T\}$ be the estimation error covariance. Now, we are in a position to state the main objectives of the addressed designed task where both ROFs and sensor saturations are taken into account within a unified framework.

*Objective 1.* We aim to design a time-varying estimator of form (11)-(12) such that there exists an upper bound of the estimation error covariance $P_{k+1|k+1}$, i.e., we are looking for a series of positive-definite matrices $\Xi_{k+1|k+1}$ satisfying

$$P_{k+1|k+1} \leq \Xi_{k+1|k+1}. \tag{13}$$

Moreover, such an upper bound $\Xi_{k+1|k+1}$ is minimized by properly designing the estimator gain $K_{k+1}$ at each time step.

*Objective 2.* We will provide a sufficient condition to verify the exponential boundedness of the estimation error in mean square sense.

To end this section, we introduce the following lemma which will be used in the proof of our main results.

**Lemma 1** *For matrices $M$, $N$, $K$ and $X$ with appropriate dimensions, the following properties hold*

$$\begin{aligned}
\frac{\partial \mathrm{tr}(MKN)}{\partial K} &= M^T N^T, \\
\frac{\partial \mathrm{tr}(MK^T N)}{\partial K} &= NM, \\
\frac{\partial \mathrm{tr}(MKNK^T X)}{\partial K} &= M^T X^T K N^T + XMKN.
\end{aligned}$$

## 3 Main Results

In this section, the upper bounds of the one-step prediction error covariance as well as estimation error covariance are obtained by employing the stochastic analysis technique. Subsequently, the desired explicit form of the estimator gain is given based on the solutions to two difference equations. At last, a sufficient condition is established to test the exponential boundedness of the estimation error in mean square sense.

### *3.1 Design of The Estimator Gain*

Let the one-step prediction error be $\tilde{x}_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k}$. Then, it follows from (8) and (11) that

$$\begin{aligned}
\tilde{x}_{k+1|k} = {} & \bar{A}_k \tilde{x}_{k|k} + [\bar{A}_{1,k} + (\alpha_k - \bar{\alpha})\bar{A}_{2,k}]x_k \\
& + g(x_k) - g(\hat{x}_{k|k}) + B_k \omega_k
\end{aligned} \tag{14}$$

where $\bar{A}_k$, $\bar{A}_{1,k}$ and $\bar{A}_{2,k}$ are defined in (10). Similarly, from (9) and (12), we have the estimation error as follows:

$$\begin{aligned}
\tilde{x}_{k+1|k+1} = {} & \tilde{x}_{k+1|k} - K_{k+1}[\sigma(C_{k+1}x_{k+1}) \\
& - C_{k+1}\hat{x}_{k+1|k} + \nu_{k+1}].
\end{aligned} \tag{15}$$

In view of (14) and (15), we can obtain the recursion forms of the one-step prediction error covariance and estimation error covariance immediately, which are shown in the following lemmas.

**Lemma 2** *The one-step prediction error covariance $P_{k+1|k} = \mathbb{E}\{\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T\}$ has the following recursion:*

$$\begin{aligned}
& P_{k+1|k} \\
= {} & \bar{A}_k P_{k|k} \bar{A}_k^T + \bar{A}_k \mathbb{E}\{\tilde{x}_{k|k} x_k^T\} \bar{A}_{1,k}^T + \bar{A}_{1,k} \mathbb{E}\{x_k \tilde{x}_{k|k}^T\} \bar{A}_k^T \\
& + \mathbb{E}\{\mathscr{A}_k + \mathscr{A}_k^T\} + \bar{A}_{1,k} \mathbb{E}\{x_k x_k^T\} \bar{A}_{1,k}^T + \mathbb{E}\{\mathscr{B}_k + \mathscr{B}_k^T\} \\
& + \bar{\alpha}(1-\bar{\alpha}) \bar{A}_{2,k} \mathbb{E}\{x_k x_k^T\} \bar{A}_{2,k}^T + \mathbb{E}\{[g(x_k) - g(\hat{x}_{k|k})] \\
& \times [g(x_k) - g(\hat{x}_{k|k})]^T\} + B_k Q_k B_k^T
\end{aligned} \tag{16}$$

*where*

$$\begin{aligned}
\mathscr{A}_k &= \bar{A}_k \tilde{x}_{k|k} [g(x_k) - g(\hat{x}_{k|k})]^T, \\
\mathscr{B}_k &= \bar{A}_{1,k} x_k [g(x_k) - g(\hat{x}_{k|k})]^T.
\end{aligned} \tag{17}$$

*Proof:* It follows from the definition of the one-step prediction error covariance that

$$\begin{aligned}
& P_{k+1|k} \\
= {} & \bar{A}_k P_{k|k} \bar{A}_k^T + \bar{A}_k \mathbb{E}\{\tilde{x}_{k|k} x_k^T\} \bar{A}_{1,k}^T + \bar{A}_{1,k} \mathbb{E}\{x_k \tilde{x}_{k|k}^T\} \bar{A}_k^T \\
& + \mathscr{P}_{1,k} + \mathscr{P}_{1,k}^T + \bar{A}_k \mathbb{E}\{\tilde{x}_{k|k} [g(x_k) - g(\hat{x}_{k|k})]^T\} \\
& + \mathbb{E}\{[g(x_k) - g(\hat{x}_{k|k})] \tilde{x}_{k|k}^T\} \bar{A}_k^T + \mathscr{P}_{2,k} + \mathscr{P}_{2,k}^T \\
& + \bar{A}_{1,k} \mathbb{E}\{x_k x_k^T\} \bar{A}_{1,k}^T + \bar{A}_{1,k} \mathbb{E}\{x_k [g(x_k) - g(\hat{x}_{k|k})]^T\} \\
& + \mathbb{E}\{[g(x_k) - g(\hat{x}_{k|k})] x_k^T\} \bar{A}_{1,k}^T + \mathscr{P}_{3,k} + \mathscr{P}_{3,k}^T \\
& + \mathscr{P}_{4,k} + \mathscr{P}_{4,k}^T + \bar{\alpha}(1-\bar{\alpha}) \bar{A}_{2,k} \mathbb{E}\{x_k x_k^T\} \bar{A}_{2,k}^T \\
& + \mathscr{P}_{5,k} + \mathscr{P}_{5,k}^T + \mathscr{P}_{6,k} + \mathscr{P}_{6,k}^T \\
& + \mathbb{E}\{[g(x_k) - g(\hat{x}_{k|k})][g(x_k) - g(\hat{x}_{k|k})]^T\} \\
& + \mathscr{P}_{7,k} + \mathscr{P}_{7,k}^T + B_k Q_k B_k^T
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{P}_{1,k} &= \mathbb{E}\{(\alpha_k - \bar{\alpha}) \bar{A}_k \tilde{x}_{k|k} x_k^T \bar{A}_{2,k}^T\}, \\
\mathscr{P}_{2,k} &= \mathbb{E}\{\bar{A}_k \tilde{x}_{k|k} \omega_k^T B_k^T\}, \\
\mathscr{P}_{3,k} &= \mathbb{E}\{(\alpha_k - \bar{\alpha}) \bar{A}_{1,k} x_k x_k^T \bar{A}_{2,k}^T\}, \\
\mathscr{P}_{4,k} &= \mathbb{E}\{\bar{A}_{1,k} x_k \omega_k^T B_k^T\}, \\
\mathscr{P}_{5,k} &= \mathbb{E}\{(\alpha_k - \bar{\alpha}) \bar{A}_{2,k} x_k [g(x_k) - g(\hat{x}_{k|k})]^T\}, \\
\mathscr{P}_{6,k} &= \mathbb{E}\{(\alpha_k - \bar{\alpha}) \bar{A}_{2,k} x_k \omega_k^T B_k^T\}, \\
\mathscr{P}_{7,k} &= \mathbb{E}\{[g(x_k) - g(\hat{x}_{k|k})] \omega_k^T B_k^T\}.
\end{aligned}$$

It is not difficult to see that the terms $\mathscr{P}_{i,k}$ $(i = 1, 2, \cdots, 7)$ are equal to zero. Then, the relationship (16) is true.

**Lemma 3** *The recursion of the estimation error covariance* $P_{k+1|k+1} = \mathbb{E}\{\tilde{x}_{k+1|k+1}\tilde{x}_{k+1|k+1}^T\}$ *can be obtained as follows:*

$$\begin{aligned}P_{k+1|k+1} =& (I - K_{k+1}C_{k+1})P_{k+1|k}(I - K_{k+1}C_{k+1})^T \\ &+\mathbb{E}\{\mathscr{C}_{k+1} + \mathscr{C}_{k+1}^T\} + \mathbb{E}\{\mathscr{D}_{k+1} + \mathscr{D}_{k+1}^T\} \\ &+K_{k+1}C_{k+1}\mathbb{E}\{x_{k+1}x_{k+1}^T\}C_{k+1}^T K_{k+1}^T \\ &+K_{k+1}\mathbb{E}\{\sigma(C_{k+1}x_{k+1})\sigma^T(C_{k+1}x_{k+1})\}K_{k+1}^T \\ &+\mathbb{E}\{\mathscr{E}_{k+1} + \mathscr{E}_{k+1}^T\} + K_{k+1}R_{k+1}K_{k+1}^T \end{aligned} \quad (18)$$

*where*

$$\begin{aligned}\mathscr{C}_{k+1} =& (I - K_{k+1}C_{k+1})\tilde{x}_{k+1|k}x_{k+1}^T C_{k+1}^T K_{k+1}^T, \\ \mathscr{D}_{k+1} =& -(I - K_{k+1}C_{k+1})\tilde{x}_{k+1|k}\sigma^T(C_{k+1}x_{k+1})K_{k+1}^T, \\ \mathscr{E}_{k+1} =& -K_{k+1}C_{k+1}x_{k+1}\sigma^T(C_{k+1}x_{k+1})K_{k+1}^T. \end{aligned} \quad (19)$$

*Proof:* Adding the zero term

$$K_{k+1}C_{k+1}x_{k+1} - K_{k+1}C_{k+1}x_{k+1}$$

to the right-hand side of (15) leads to

$$\begin{aligned}\tilde{x}_{k+1|k+1} =& (I - K_{k+1}C_{k+1})\tilde{x}_{k+1|k} + K_{k+1}C_{k+1}x_{k+1} \\ &-K_{k+1}\sigma(C_{k+1}x_{k+1}) - K_{k+1}\nu_{k+1}. \end{aligned} \quad (20)$$

Based on (20), we arrive at

$$\begin{aligned}&P_{k+1|k+1} \\ =& (I - K_{k+1}C_{k+1})P_{k+1|k}(I - K_{k+1}C_{k+1})^T \\ &+(I - K_{k+1}C_{k+1})\mathbb{E}\{\tilde{x}_{k+1|k}x_{k+1}^T\}C_{k+1}^T K_{k+1}^T \\ &+K_{k+1}C_{k+1}\mathbb{E}\{x_{k+1}\tilde{x}_{k+1|k}^T\}(I - K_{k+1}C_{k+1})^T \\ &-(I - K_{k+1}C_{k+1})\mathbb{E}\{\tilde{x}_{k+1|k}\sigma^T(C_{k+1}x_{k+1})\}K_{k+1}^T \\ &-K_{k+1}\mathbb{E}\{\sigma(C_{k+1}x_{k+1})\tilde{x}_{k+1|k}^T\}(I - K_{k+1}C_{k+1})^T \\ &-\mathscr{Q}_{1,k+1} - \mathscr{Q}_{1,k+1}^T + K_{k+1}C_{k+1}\mathbb{E}\{x_{k+1}x_{k+1}^T\}C_{k+1}^T \\ &\times K_{k+1}^T - K_{k+1}C_{k+1}\mathbb{E}\{x_{k+1}\sigma^T(C_{k+1}x_{k+1})\}K_{k+1}^T \\ &-K_{k+1}\mathbb{E}\{\sigma(C_{k+1}x_{k+1})x_{k+1}^T\}C_{k+1}^T K_{k+1}^T - \mathscr{Q}_{2,k+1} \\ &-\mathscr{Q}_{2,k+1}^T + K_{k+1}\mathbb{E}\{\sigma(C_{k+1}x_{k+1})\sigma^T(C_{k+1}x_{k+1})\} \\ &\times K_{k+1}^T + \mathscr{Q}_{3,k+1} + \mathscr{Q}_{3,k+1}^T + K_{k+1}R_{k+1}K_{k+1}^T \end{aligned}$$

where

$$\begin{aligned}\mathscr{Q}_{1,k+1} =& \mathbb{E}\{(I - K_{k+1}C_{k+1})\tilde{x}_{k+1|k}\nu_{k+1}^T K_{k+1}^T\}, \\ \mathscr{Q}_{2,k+1} =& \mathbb{E}\{K_{k+1}C_{k+1}x_{k+1}\nu_{k+1}^T K_{k+1}^T\}, \\ \mathscr{Q}_{3,k+1} =& \mathbb{E}\{K_{k+1}\sigma(C_{k+1}x_{k+1})\nu_{k+1}^T K_{k+1}^T\}. \end{aligned}$$

Subsequently, it is easy to see that the terms $\mathscr{Q}_{i,k+1}$ $(i = 1, 2, 3)$ are equal to zero. Consequently, it can be concluded that (18) is true, and this ends the proof of this Lemma.

**Remark 4** *It should be pointed out that there exist some unknown terms in Lemmas 2–3 due to the existence of parameter uncertainties, nonlinearities and sensor saturations. Then, the exact value of the one-step prediction error covariance cannot be obtained. Hence, it is literally impossible to obtain the accurate value of the estimation error covariance and then it is difficult to quantify the achievable performance requirements for addressed joint estimation problem. In the sequel, we aim to look for an upper bound of the estimation error covariance and minimize such an upper bound at each sampling instant by properly designing the estimator gain with the help of the recursive difference equation approach.*

**Theorem 1** *For positive scalars* $\gamma_{k,i}$ $(i = 1, \cdots, 4)$ *and* $\gamma_{k+1,j}$ $(j = 5, \cdots, 8)$*, under the initial condition* $\Xi_{0|0} = P_{0|0}$*, assume that the following two difference equations have solutions* $\Xi_{k+1|k}$ *and* $\Xi_{k+1|k+1}$*:*

$$\begin{aligned}\Xi_{k+1|k} =& (1 + \gamma_{k,1} + \gamma_{k,2})\bar{A}_k\Xi_{k|k}\bar{A}_k^T + (1 + \gamma_{k,1}^{-1} + \gamma_{k,3}) \\ &\times\mathrm{tr}\left(\bar{N}_k\Theta_k\bar{N}_k^T\right)\bar{M}_k\bar{M}_k^T + \bar{\alpha}(1 - \bar{\alpha})\bar{A}_{2,k}\Theta_k\bar{A}_{2,k}^T \\ &+(1 + \gamma_{k,2}^{-1} + \gamma_{k,3}^{-1})\bar{G}\Xi_{k|k}\bar{G}^T + B_kQ_kB_k^T, \end{aligned} \quad (21)$$

*and*

$$\begin{aligned}\Xi_{k+1|k+1} =& (1 + \gamma_{k+1,5} + \gamma_{k+1,6})(I - K_{k+1}C_{k+1})\Xi_{k+1|k} \\ &\times(I - K_{k+1}C_{k+1})^T + (1 + \gamma_{k+1,5}^{-1} + \gamma_{k+1,7}) \\ &\times K_{k+1}C_{k+1}\Omega_{k+1}C_{k+1}^T K_{k+1}^T + K_{k+1} \\ &\times[\bar{\varrho}(1 + \gamma_{k+1,6}^{-1} + \gamma_{k+1,7}^{-1})I + R_{k+1}]K_{k+1}^T \end{aligned} \quad (22)$$

*where*

$$\begin{aligned}\bar{\varrho} =& \sum_{i=1}^m \varrho_i^2, \quad \bar{M}_k^T = \left[\begin{array}{cc}\vec{M}_k^T & 0\end{array}\right], \\ \bar{N}_k =& \left[\begin{array}{cc}\vec{N}_k & 0\end{array}\right], \quad \bar{G} = \mathrm{diag}\{G, 0\}, \\ \Theta_k =& (1 + \gamma_{k,4})\Xi_{k|k} + (1 + \gamma_{k,4}^{-1})\hat{x}_{k|k}\hat{x}_{k|k}^T, \\ \Omega_{k+1} =& (1 + \gamma_{k+1,8})\Xi_{k+1|k} + (1 + \gamma_{k+1,8}^{-1})\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T. \end{aligned} \quad (23)$$

*Then, the matrix* $\Xi_{k+1|k+1}$ *is an upper bound of* $P_{k+1|k+1}$*, i.e.,*

$$P_{k+1|k+1} \le \Xi_{k+1|k+1}. \quad (24)$$

*Moreover, such an upper bound* $\Xi_{k+1|k+1}$ *can be minimized at each time step by using the following estimator gain*

$$K_{k+1} = (1 + \gamma_{k+1,5} + \gamma_{k+1,6})\Xi_{k+1|k}C_{k+1}^T\mathscr{K}_{k+1}^{-1} \quad (25)$$

*where*

$$\begin{aligned}\mathscr{K}_{k+1} =& (1 + \gamma_{k+1,5} + \gamma_{k+1,6})C_{k+1}\Xi_{k+1|k}C_{k+1}^T \\ &+(1 + \gamma_{k+1,5}^{-1} + \gamma_{k+1,7})C_{k+1}\Omega_{k+1}C_{k+1}^T \\ &+\bar{\varrho}(1 + \gamma_{k+1,6}^{-1} + \gamma_{k+1,7}^{-1})I + R_{k+1}. \end{aligned} \quad (26)$$

*Proof:* Firstly, we will handle the unknown terms of the right-hand side of (16) and then derive the upper bound of the one-step prediction error covariance. From the following elementary inequality

$$ab^T + ba^T \le \gamma aa^T + \gamma^{-1}bb^T \quad (27)$$

where $\gamma$ is a positive scalar and $a$, $b$ are vectors of appropriate dimensions, one has

$$\bar{A}_k\mathbb{E}\{\tilde{x}_{k|k}x_k^T\}\bar{A}_{1,k}^T + \bar{A}_{1,k}\mathbb{E}\{x_k\tilde{x}_{k|k}^T\}\bar{A}_k^T$$
$$\le \gamma_{k,1}\bar{A}_kP_{k|k}\bar{A}_k^T + \gamma_{k,1}^{-1}\bar{A}_{1,k}\mathbb{E}\{x_kx_k^T\}\bar{A}_{1,k}^T, \quad (28)$$

where $\gamma_{k,1}$ is a positive scalar. Similarly, it follows from (27) that

$$\begin{aligned}&\mathbb{E}\{\mathscr{A}_k + \mathscr{A}_k^T\}\\ &\le \gamma_{k,2}\bar{A}_kP_{k|k}\bar{A}_k^T + \gamma_{k,2}^{-1}\mathbb{E}\{[g(x_k) - g(\hat{x}_{k|k})]\\ &\quad\times[g(x_k) - g(\hat{x}_{k|k})]^T\}, \quad (29)\\ &\mathbb{E}\{\mathscr{B}_k + \mathscr{B}_k^T\}\\ &\le \gamma_{k,3}\bar{A}_{1,k}\mathbb{E}\{x_kx_k^T\}\bar{A}_{1,k}^T + \gamma_{k,3}^{-1}\mathbb{E}\{[g(x_k) - g(\hat{x}_{k|k})]\\ &\quad\times[g(x_k) - g(\hat{x}_{k|k})]^T\}, \quad (30)\end{aligned}$$

where $\gamma_{k,2}$ and $\gamma_{k,3}$ are positive scalars. Substituting (28)–(30) into (16) leads to

$$\begin{aligned}P_{k+1|k} \le{}& (1 + \gamma_{k,1} + \gamma_{k,2})\bar{A}_kP_{k|k}\bar{A}_k^T + (1 + \gamma_{k,1}^{-1} + \gamma_{k,3})\\ &\times\bar{A}_{1,k}\mathbb{E}\{x_kx_k^T\}\bar{A}_{1,k}^T + (1 + \gamma_{k,2}^{-1} + \gamma_{k,3}^{-1})\\ &\times\mathbb{E}\{[g(x_k) - g(\hat{x}_{k|k})][g(x_k) - g(\hat{x}_{k|k})]^T\}\\ &+\bar{\alpha}(1 - \bar{\alpha})\bar{A}_{2,k}\mathbb{E}\{x_kx_k^T\}\bar{A}_{2,k}^T + B_kQ_kB_k^T. \quad (31)\end{aligned}$$

Next, from (27), we have

$$\begin{aligned}&\mathbb{E}\{x_kx_k^T\}\\ &= \mathbb{E}\{(\tilde{x}_{k|k} + \hat{x}_{k|k})(\tilde{x}_{k|k} + \hat{x}_{k|k})^T\}\\ &\le (1 + \gamma_{k,4})P_{k|k} + (1 + \gamma_{k,4}^{-1})\hat{x}_{k|k}\hat{x}_{k|k}^T, \quad (32)\end{aligned}$$

where $\gamma_{k,4}$ is a positive scalar. Then, it follows from (3) and (32) that

$$\begin{aligned}&\bar{A}_{1,k}\mathbb{E}\{x_kx_k^T\}\bar{A}_{1,k}^T\\ &= \bar{M}_k\mathcal{F}_k\bar{N}_k\left[(1 + \gamma_{k,4})P_{k|k} + (1 + \gamma_{k,4}^{-1})\hat{x}_{k|k}\hat{x}_{k|k}^T\right]\\ &\quad\times\bar{N}_k^T\mathcal{F}_k^T\bar{M}_k^T\\ &\le \mathrm{tr}\left\{\bar{N}_k\left[(1 + \gamma_{k,4})P_{k|k} + (1 + \gamma_{k,4}^{-1})\hat{x}_{k|k}\hat{x}_{k|k}^T\right]\bar{N}_k^T\right\}\\ &\quad\times\bar{M}_k\bar{M}_k^T, \quad (33)\end{aligned}$$

where $\bar{M}_k$ and $\bar{N}_k$ are defined as in (23). In view of (4), it follows that

$$\begin{aligned}&\mathbb{E}\{[g(x_k) - g(\hat{x}_{k|k})][g(x_k) - g(\hat{x}_{k|k})]^T\}\\ &\le \mathbb{E}\{\bar{G}\tilde{x}_{k|k}\tilde{x}_{k|k}^T\bar{G}^T\} = \bar{G}P_{k|k}\bar{G}^T \quad (34)\end{aligned}$$

where $\bar{G}$ is defined as in (23). Thus, according to (31)–(34), we can obtain the following inequality

$$\begin{aligned}&P_{k+1|k}\\ &\le (1 + \gamma_{k,1} + \gamma_{k,2})\bar{A}_kP_{k|k}\bar{A}_k^T + (1 + \gamma_{k,1}^{-1} + \gamma_{k,3})\\ &\quad\times\mathrm{tr}\left\{\bar{N}_k\left[(1 + \gamma_{k,4})P_{k|k} + (1 + \gamma_{k,4}^{-1})\hat{x}_{k|k}\hat{x}_{k|k}^T\right]\bar{N}_k^T\right\}\\ &\quad\times\bar{M}_k\bar{M}_k^T + \bar{\alpha}(1 - \bar{\alpha})\bar{A}_{2,k}\big[(1 + \gamma_{k,4})P_{k|k}\\ &\quad+(1 + \gamma_{k,4}^{-1})\hat{x}_{k|k}\hat{x}_{k|k}^T\big]\bar{A}_{2,k}^T + (1 + \gamma_{k,2}^{-1} + \gamma_{k,3}^{-1})\\ &\quad\times\bar{G}P_{k|k}\bar{G}^T + B_kQ_kB_k^T. \quad (35)\end{aligned}$$

Secondly, let us deal with the unknown terms of the right-hand side of (18). By using the inequality (27) again, we have

$$\begin{aligned}&\mathbb{E}\{\mathscr{C}_{k+1} + \mathscr{C}_{k+1}^T\}\\ &\le \gamma_{k+1,5}(I - K_{k+1}C_{k+1})P_{k+1|k}(I - K_{k+1}C_{k+1})^T\\ &\quad+\gamma_{k+1,5}^{-1}K_{k+1}C_{k+1}\mathbb{E}\{x_{k+1}x_{k+1}^T\}C_{k+1}^TK_{k+1}^T, \quad (36)\\ &\mathbb{E}\{\mathscr{D}_{k+1} + \mathscr{D}_{k+1}^T\}\\ &\le \gamma_{k+1,6}(I - K_{k+1}C_{k+1})P_{k+1|k}(I - K_{k+1}C_{k+1})^T\\ &\quad+\gamma_{k+1,6}^{-1}K_{k+1}\mathbb{E}\{\sigma(C_{k+1}x_{k+1})\sigma^T(C_{k+1}x_{k+1})\}K_{k+1}^T, \quad (37)\\ &\mathbb{E}\{\mathscr{E}_{k+1} + \mathscr{E}_{k+1}^T\}\\ &\le \gamma_{k+1,7}K_{k+1}C_{k+1}\mathbb{E}\{x_{k+1}x_{k+1}^T\}C_{k+1}^TK_{k+1}^T + \gamma_{k+1,7}^{-1}\\ &\quad\times K_{k+1}\mathbb{E}\{\sigma(C_{k+1}x_{k+1})\sigma^T(C_{k+1}x_{k+1})\}K_{k+1}^T \quad (38)\end{aligned}$$

where $\gamma_{k+1,i}$ $(i = 5, 6, 7)$ are positive scalars. Similar to (32), it follows that

$$\begin{aligned}&\mathbb{E}\{x_{k+1}x_{k+1}^T\}\\ &\le (1 + \gamma_{k+1,8})P_{k+1|k} + (1 + \gamma_{k+1,8}^{-1})\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T \quad (39)\end{aligned}$$

with $\gamma_{k+1,8}$ being a positive scalar. Moreover, one has from (7) that

$$\mathbb{E}\{\sigma(C_{k+1}x_{k+1})\sigma^T(C_{k+1}x_{k+1})\} \le \bar{\varrho}I \quad (40)$$

where $\bar{\varrho}$ is defined as in (23). Substituting (36)–(40) into (18) yields

$$\begin{aligned}P_{k+1|k+1} \le{}& (1 + \gamma_{k+1,5} + \gamma_{k+1,6})(I - K_{k+1}C_{k+1})P_{k+1|k}\\ &\times(I - K_{k+1}C_{k+1})^T + (1 + \gamma_{k+1,5}^{-1} + \gamma_{k+1,7})\\ &\times K_{k+1}C_{k+1}\bar{\Omega}_{k+1}C_{k+1}^TK_{k+1}^T + K_{k+1}\\ &\times[\bar{\varrho}(1 + \gamma_{k+1,6}^{-1} + \gamma_{k+1,7}^{-1})I + R_{k+1}]K_{k+1}^T \quad (41)\end{aligned}$$

where

$$\bar{\Omega}_{k+1} = (1 + \gamma_{k+1,8})P_{k+1|k} + (1 + \gamma_{k+1,8}^{-1})\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T.$$

Then, based on the mathematical induction approach, it is not difficult to verify that

$$P_{k+1|k+1} \le \Xi_{k+1|k+1}. \quad (42)$$

Finally, we are ready to derive the estimator gain which can minimize the obtained upper bound $\Xi_{k+1|k+1}$. According to Lemma 1, taking the partial derivative of the trace of (22) with respect to $K_{k+1}$ and letting the derivative be zero, we obtain

$$\begin{aligned}&\frac{\partial\mathrm{tr}(\Xi_{k+1|k+1})}{\partial K_{k+1}}\\ &= -2(1 + \gamma_{k+1,5} + \gamma_{k+1,6})(I - K_{k+1}C_{k+1})\Xi_{k+1|k}C_{k+1}^T\\ &\quad+2K_{k+1}\Big[(1 + \gamma_{k+1,5}^{-1} + \gamma_{k+1,7})C_{k+1}\Omega_{k+1}C_{k+1}^T\\ &\quad+\bar{\varrho}(1 + \gamma_{k+1,6}^{-1} + \gamma_{k+1,7}^{-1})I + R_{k+1}\Big]\end{aligned}$$

$$= 0. \tag{43}$$

Based on (43) and through tedious algebraic manipulations, the estimator gain can be determined as in (25). Therefore, the proof of this theorem is complete.

So far, we have obtained an upper bound of the estimation error covariance and minimized such an upper bound by designing proper estimator gain. In terms of the solutions to two difference equations, the explicit form of the estimator gain has also been proposed. It is worthwhile to note that the developed estimation scheme can estimate the systems states and fault simultaneously and, moreover, the proposed estimation algorithm is of a recursive feature suitable for online applications.

**Remark 5** *To deal with the computational complexity of the developed joint estimation algorithm, we recall that the variable dimensions can be seen from $x_k \in \mathbb{R}^n$ and $f_k \in \mathbb{R}^f$. It is not difficult to obtain the overall computational complexity of the proposed estimation algorithm as $O((n+n_f)^3)$, which depends on the variable dimension. It can be easily seen that the computational burden is mainly caused by the basic mathematical operations. Fortunately, many methods can be utilized to improve the computational efficiency of mathematical operations in the areas of computational mathematics and optimization. On the other hand, in case that the robustness of the developed estimation algorithm becomes a concern, we can introduce the robustness criterion in both state estimation and fault estimation simultaneously with respect to the uncertainties, which constitutes one of future research directions.*

In what follows, we are ready to conduct the performance analysis of the developed estimation algorithm and provide a sufficient condition to ensure that the estimation error is exponentially bounded in mean square sense.

### *3.2 Performance Analysis*

To facilitate further developments, let us introduce the following definition regarding the boundedness of a stochastic process.

**Definition 1** *[26] The stochastic process $\zeta_k$ is said to be exponentially bounded in mean square, if there are real numbers $\eta > 0$, $\nu > 0$ and $0 < \vartheta < 1$ such that*

$$\mathbb{E}\{\|\zeta_k\|^2\} \le \eta \mathbb{E}\{\|\zeta_0\|^2\}\vartheta^k + \nu \tag{44}$$

*holds for every time step $k \ge 0$.*

Based on Definition 1, a sufficient criterion is given in the following theorem to verify the exponential boundedness of the estimation error in mean square sense.

**Theorem 2** *Consider the time-varying nonlinear systems (1)-(2) with the designed estimator as in (11)-(12). Assume that there exist positive real scalars $\bar{a}$, $\underline{b}$, $\bar{b}$, $\underline{c}$, $\bar{c}$, $\bar{f}$, $\bar{m}$, $\bar{n}$, $\bar{g}$, $\underline{q}_1$, $\bar{q}_1$, $\bar{q}_2$, $\bar{\vartheta}_1$, $\bar{\vartheta}_2$ and $\bar{\vartheta}_3$ such that the following conditions*

$$\begin{aligned}
&\|\bar{A}_k\| \le \bar{a}, \ \ \underline{b}I \le \vec{B}_k\vec{B}_k^T \le \bar{b}I, \ \ \underline{c} \le \|\vec{C}_k\| \le \bar{c},\\
&\|\vec{F}_k\| \le \bar{f}, \ \ \|\vec{M}_k\| \le \bar{m}, \ \ \|\vec{N}_k\| \le \bar{n}, \ \ \|G\| \le \bar{g},\\
&\underline{q}_1 I \le Q_k \le \bar{q}_1 I, \ \ R_k \le \bar{q}_2 I, \ \ \mathrm{tr}(\Theta_k) \le \bar{\vartheta}_1,\\
&\mathrm{tr}(\Xi_{k|k}) \le \bar{\vartheta}_2, \ \ \mathrm{tr}(\Omega_{k+1}) \le \bar{\vartheta}_3,
\end{aligned}$$

$$\rho = \bar{a}^2\left(1 + \frac{\bar{c}^2}{\underline{c}^2}\right)^2 < 1, \tag{45}$$

*hold, then the estimation error is exponentially bounded in mean square sense.*

*Proof:* Substituting (14) into (20) yields

$$\tilde{x}_{k+1|k+1} = \Lambda_{k+1}\bar{A}_k\tilde{x}_{k|k} + r_{k+1} + s_{k+1} \tag{46}$$

where

$$\begin{aligned}
\Lambda_{k+1} &= I - K_{k+1}C_{k+1},\\
r_{k+1} &= \Lambda_{k+1}\bar{A}_{1,k}x_k + \Lambda_{k+1}[g(x_k) - g(\hat{x}_{k|k})]\\
&\quad + K_{k+1}[C_{k+1}x_{k+1} - \sigma(C_{k+1}x_{k+1})],\\
s_{k+1} &= (\alpha_k - \bar{\alpha})\Lambda_{k+1}\bar{A}_{2,k}x_k + \Lambda_{k+1}B_k\omega_k - K_{k+1}\nu_{k+1}.
\end{aligned} \tag{47}$$

From (25) and (26), we have

$$\begin{aligned}
&\|K_{k+1}\|\\
&= \|(1 + \gamma_{k+1,5} + \gamma_{k+1,6})\Xi_{k+1|k}C_{k+1}^T\mathscr{K}_{k+1}^{-1}\|\\
&< \|(1 + \gamma_{k+1,5} + \gamma_{k+1,6})\Xi_{k+1|k}C_{k+1}^T\\
&\quad \times\big[(1 + \gamma_{k+1,5} + \gamma_{k+1,6})C_{k+1}\Xi_{k+1|k}C_{k+1}^T\big]^{-1}\|\\
&\le \frac{\bar{c}}{\underline{c}^2} \triangleq \bar{k}.
\end{aligned} \tag{48}$$

Similarly, it is not difficult to verify that

$$\|\Lambda_{k+1}\| = \|I - K_{k+1}C_{k+1}\| \le 1 + \frac{\bar{c}^2}{\underline{c}^2} \triangleq \bar{\lambda}. \tag{49}$$

Next, it follows from (27) and (45) that

$$\begin{aligned}
&\mathbb{E}\{r_{k+1}^T r_{k+1}\}\\
&\le (1 + \varsigma_1 + \varsigma_2)\mathbb{E}\{x_k^T\bar{A}_{1,k}^T\Lambda_{k+1}^T\Lambda_{k+1}\bar{A}_{1,k}x_k\}\\
&\quad + (1 + \varsigma_1^{-1} + \varsigma_3)\mathbb{E}\{[g(x_k) - g(\hat{x}_{k|k})]^T\Lambda_{k+1}^T\Lambda_{k+1}\\
&\quad \times[g(x_k) - g(\hat{x}_{k|k})]\} + (1 + \varsigma_2^{-1} + \varsigma_3^{-1})\\
&\quad \times\mathbb{E}\{[C_{k+1}x_{k+1} - \sigma(C_{k+1}x_{k+1})]^T K_{k+1}^T K_{k+1}\\
&\quad \times[C_{k+1}x_{k+1} - \sigma(C_{k+1}x_{k+1})]\}\\
&\le (1 + \varsigma_1 + \varsigma_2)\mathrm{tr}\big(\mathbb{E}\{x_k x_k^T\}\bar{N}_k^T\mathcal{F}_k^T\bar{M}_k^T\Lambda_{k+1}^T\Lambda_{k+1}\\
&\quad \times\bar{M}_k\mathcal{F}_k\bar{N}_k\big) + (1 + \varsigma_1^{-1} + \varsigma_3)\mathrm{tr}(\bar{G}\Xi_{k|k}\bar{G}^T\Lambda_{k+1}^T\Lambda_{k+1})\\
&\quad + (1 + \varsigma_2^{-1} + \varsigma_3^{-1})\bigg\{(1 + \varsigma_4)\mathrm{tr}(\mathbb{E}\{x_{k+1}x_{k+1}^T\}C_{k+1}^T\\
&\quad \times K_{k+1}^T K_{k+1}C_{k+1}) + (1 + \varsigma_4^{-1})\mathrm{tr}\big[\mathbb{E}\{\sigma(C_{k+1}x_{k+1})\\
&\quad \times\sigma^T(C_{k+1}x_{k+1})\}K_{k+1}^T K_{k+1}\big]\bigg\}\\
&\le (1 + \varsigma_1 + \varsigma_2)\bar{m}^2\bar{\lambda}^2\bar{n}^2\bar{\vartheta}_1 + (1 + \varsigma_1^{-1} + \varsigma_3)\bar{\lambda}^2\bar{g}^2\bar{\vartheta}_2\\
&\quad + (1 + \varsigma_2^{-1} + \varsigma_3^{-1})\big[(1 + \varsigma_4)\bar{c}^2\bar{k}^2\bar{\vartheta}_3\\
&\quad + (1 + \varsigma_4^{-1})\bar{k}^2\bar{\varrho}m\big] \triangleq \bar{r},
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
&\mathbb{E}\{s_{k+1}^T s_{k+1}\}\\
&= \bar{\alpha}(1 - \bar{\alpha})\mathbb{E}\{x_k^T\bar{A}_{2,k}^T\Lambda_{k+1}^T\Lambda_{k+1}\bar{A}_{2,k}x_k\}
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\{\omega_k^T B_k^T \Lambda_{k+1}^T \Lambda_{k+1} B_k \omega_k\} \\
&+\mathbb{E}\{\nu_{k+1}^T K_{k+1}^T K_{k+1} \nu_{k+1}\} \\
&\leq \bar{\alpha}(1-\bar{\alpha})\bar{\lambda}^2 \bar{f}^2 \bar{\vartheta}_1 + \bar{q}_1 n \bar{b} \bar{\lambda}^2 + \bar{q}_2 m \bar{k}^2 \triangleq \bar{s},
\end{aligned} \tag{51}
$$

where $\varsigma_i$ $(i = 1, 2, 3, 4)$ are positive scalars.

Subsequently, consider the following iterative matrix equation with respect to $\Psi_k$:

$$
\Psi_{k+1} = \Lambda_{k+1} \bar{A}_k \Psi_k \bar{A}_k^T \Lambda_{k+1}^T + B_k Q_k B_k^T + \varepsilon I \tag{52}
$$

where $\Psi_0 = B_0 Q_0 B_0^T + \varepsilon I$ and $\varepsilon > 0$ is a scalar. Then, according to the above iterative matrix equation, it suffices to see that

$$
\begin{aligned}
\|\Psi_{k+1}\| &\leq \|\Lambda_{k+1}\|^2 \|\bar{A}_k\|^2 \|\Psi_k\| + \|B_k Q_k B_k^T\| + \|\varepsilon I\| \\
&\leq \rho \|\Psi_k\| + \bar{b}\bar{q}_1 + \varepsilon
\end{aligned} \tag{53}
$$

where $\rho$ is defined as in (45). Furthermore, the following inequality can be obtained directly

$$
\|\Psi_k\| \leq \rho^k \|\Psi_0\| + (\bar{b}\bar{q} + \varepsilon) \sum_{i=0}^{k-1} \rho^i. \tag{54}
$$

It follows from $\rho < 1$ that

$$
\|\Psi_k\| < \|\Psi_0\| + (\bar{b}\bar{q} + \varepsilon) \sum_{i=0}^{\infty} \rho^i = \|\Psi_0\| + \frac{\bar{b}\bar{q} + \varepsilon}{1 - \rho} \tag{55}
$$

On the other hand, we can see that

$$
\Psi_k \geq \varepsilon I. \tag{56}
$$

In view of (55) and (56), there exist two positive scalars $\bar{\psi}$ and $\underline{\psi}$ such that $\underline{\psi} I \leq \Psi_k \leq \bar{\psi} I$ is true for all $k \geq 0$.

Let $V_k(\tilde{x}_{k|k}) = \tilde{x}_{k|k}^T \Psi_k^{-1} \tilde{x}_{k|k}$. For a positive scalar $\delta$, it follows from (27) and (46) that

$$
\begin{aligned}
&\mathbb{E}\{V_{k+1}(\tilde{x}_{k+1|k+1}) | \tilde{x}_{k|k}\} - (1+\delta) V_k(\tilde{x}_{k|k}) \\
&= \mathbb{E}\Big\{ \left[\Lambda_{k+1} \bar{A}_k \tilde{x}_{k|k} + r_{k+1} + s_{k+1}\right]^T \Psi_{k+1}^{-1} \\
&\quad \times \left[\Lambda_{k+1} \bar{A}_k \tilde{x}_{k|k} + r_{k+1} + s_{k+1}\right] \Big\} - (1+\delta) \tilde{x}_{k|k}^T \Psi_k^{-1} \tilde{x}_{k|k} \\
&= \mathbb{E}\left\{ \tilde{x}_{k|k}^T [\bar{A}_k^T \Lambda_{k+1}^T \Psi_{k+1}^{-1} \Lambda_{k+1} \bar{A}_k - (1+\delta) \Psi_k^{-1}] \tilde{x}_{k|k} \right\} \\
&\quad + 2\mathbb{E}\left\{ \tilde{x}_{k|k}^T \bar{A}_k^T \Lambda_{k+1}^T \Psi_{k+1}^{-1} r_{k+1} \right\} \\
&\quad + \mathbb{E}\left\{ r_{k+1}^T \Psi_{k+1}^{-1} r_{k+1} \right\} + \mathbb{E}\left\{ s_{k+1}^T \Psi_{k+1}^{-1} s_{k+1} \right\} \\
&\leq (1+\delta) \mathbb{E}\left\{ \tilde{x}_{k|k}^T [\bar{A}_k^T \Lambda_{k+1}^T \Psi_{k+1}^{-1} \Lambda_{k+1} \bar{A}_k - \Psi_k^{-1}] \tilde{x}_{k|k} \right\} \\
&\quad + (1+\delta^{-1}) \mathbb{E}\left\{ r_{k+1}^T \Psi_{k+1}^{-1} r_{k+1} \right\} \\
&\quad + \mathbb{E}\left\{ s_{k+1}^T \Psi_{k+1}^{-1} s_{k+1} \right\}.
\end{aligned} \tag{57}
$$

According to the definition of matrix $\Psi_{k+1}$ and employing the matrix inversion lemma, we have

$$
\begin{aligned}
&\bar{A}_k^T \Lambda_{k+1}^T \Psi_{k+1}^{-1} \Lambda_{k+1} \bar{A}_k - \Psi_k^{-1} \\
&= \bar{A}_k^T \Lambda_{k+1}^T \left( \Lambda_{k+1} \bar{A}_k \Psi_k \bar{A}_k^T \Lambda_{k+1}^T + B_k Q_k B_k^T + \varepsilon I \right)^{-1} \\
&\quad \times \Lambda_{k+1} \bar{A}_k - \Psi_k^{-1} \\
&= -\left[ \Psi_k + \Psi_k \bar{A}_k^T \Lambda_{k+1}^T \left( B_k Q_k B_k^T + \varepsilon I \right)^{-1} \Lambda_{k+1} \bar{A}_k \Psi_k \right]^{-1} \\
&= -\left[ I + \bar{A}_k^T \Lambda_{k+1}^T \left( B_k Q_k B_k^T + \varepsilon I \right)^{-1} \Lambda_{k+1} \bar{A}_k \Psi_k \right]^{-1} \Psi_k^{-1} \\
&\leq -\left( 1 + \frac{\bar{a}^2 \bar{\lambda}^2 \bar{\psi}}{\underline{q}_1 \underline{b}} \right)^{-1} \Psi_k^{-1}
\end{aligned} \tag{58}
$$

Substituting (58) into (57) implies

$$
\begin{aligned}
&\mathbb{E}\{V_{k+1}(\tilde{x}_{k+1|k+1}) | \tilde{x}_{k|k}\} - (1+\delta) V_k(\tilde{x}_{k|k}) \\
&\leq -(1+\delta) \left( 1 + \frac{\bar{a}^2 \bar{\lambda}^2 \bar{\psi}}{\underline{q}_1 \underline{b}} \right)^{-1} V_k(\tilde{x}_{k|k}) + \kappa
\end{aligned} \tag{59}
$$

with $\kappa = (1+\delta^{-1}) \frac{\bar{r}^2}{\underline{\psi}} + \frac{\bar{s}^2}{\underline{\psi}}$. Then, based on (59), we obtain

$$
\mathbb{E}\{V_{k+1}(\tilde{x}_{k+1|k+1}) | \tilde{x}_{k|k}\} \leq \chi V_k(\tilde{x}_{k|k}) + \kappa \tag{60}
$$

where $\chi = (1+\delta) \left[ 1 - \left( 1 + \frac{\bar{a}^2 \bar{\lambda}^2 \bar{\psi}}{\underline{q}_1 \underline{b}} \right)^{-1} \right]$. It is not difficult to see that $\chi \in (0, 1)$ for some $\delta > 0$. Furthermore, we have

$$
\begin{aligned}
&\mathbb{E}\{\|\tilde{x}_{k+1|k+1}\|^2\} \\
&\leq \frac{\bar{\psi}}{\underline{\psi}} \mathbb{E}\{\|\tilde{x}_{0|0}\|^2\} \chi^{k+1} + \kappa \bar{\psi} \sum_{i=0}^{k} \chi^i \\
&\leq \frac{\bar{\psi}}{\underline{\psi}} \mathbb{E}\{\|\tilde{x}_{0|0}\|^2\} \chi^{k+1} + \kappa \bar{\psi} \sum_{i=0}^{\infty} \chi^i \\
&= \frac{\bar{\psi}}{\underline{\psi}} \mathbb{E}\{\|\tilde{x}_{0|0}\|^2\} \chi^{k+1} + \frac{\kappa \bar{\psi}}{1 - \chi}.
\end{aligned} \tag{61}
$$

According to Definition 1, it can be concluded that the stochastic process $\tilde{x}_{k|k}$ is exponentially bounded in mean square sense.

On the other hand, the estimation is biased because of the introduced nonlinearity, sensor saturations and randomly occurring faults. As such, it is often desirable to estimate the upper bound of the bias. Since

$$
\begin{aligned}
&\mathbb{E}\{(\tilde{x}_{k+1|k+1} - \mathbb{E}\{\tilde{x}_{k+1|k+1}\})^T (\tilde{x}_{k+1|k+1} - \mathbb{E}\{\tilde{x}_{k+1|k+1}\})\} \\
&= \mathbb{E}\{\|\tilde{x}_{k+1|k+1}\|^2\} - \|\mathbb{E}\{\tilde{x}_{k+1|k+1}\}\|^2 \\
&\geq 0,
\end{aligned}
$$

we have the following bias estimate

$$
\begin{aligned}
\|\mathbb{E}\{\tilde{x}_{k+1|k+1}\}\| &\leq \sqrt{\mathbb{E}\{\|\tilde{x}_{k+1|k+1}\|^2\}} \\
&\leq \sqrt{\frac{\bar{\psi}}{\underline{\psi}} \mathbb{E}\{\|\tilde{x}_{0|0}\|^2\} \chi^{k+1} + \frac{\kappa \bar{\psi}}{1 - \chi}}.
\end{aligned}
$$

The proof of this theorem is complete now.

**Remark 6** *Up to now, we have addressed the joint estimation problem for a class of uncertain time-varying nonlinear systems with ROFs and sensor saturations.*

*A new estimation scheme has been provided to estimate the systems states and fault in a unified framework and the performance analysis of the estimation algorithm has been conducted to reveal the boundedness behaviour of the estimation error. In the time-varying stochastic model addressed in the paper, there are four main aspects that complicate the design of estimation algorithm, i.e., parameter uncertainties, nonlinearity, ROFs and sensor saturations. It is worthwhile to mention that, with the designed time-varying estimator of form (11)-(12), the proposed joint estimation algorithm has the following advantages: 1) the time-varying estimator structure is simple and of a recursive form, hence the presented estimation algorithm can be easily implemented in real time; and 2) the effects from the parameter uncertainties, nonlinearity, ROFs and sensor saturations are explicitly reflected in the algorithms. To be specific, matrices* $\vec{M}_k$ *and* $\vec{N}_k$ *quantify the parameter uncertainties, the matrix* $G$ *corresponds to the nonlinearity, the occurrence probability* $\bar{\alpha}$ *is there for the ROFs, and the scalar* $\varrho_i$ $(i = 1, 2, \cdots, m)$ *accounts for the saturation level. Compared with existing methods, the major advantage of the paper lies in that the effects from the mentioned four facts onto the algorithm performance has been examined and revealed in a unified framework with provided analysis criterion.*

**Remark 7** *It should be noted that the additive noises, nonlinearity, sensor saturations and fault lead to the derivation of the possible equilibrium. Therefore, we aim to consider the exponential boundedness in mean square (rather than the stability) of the estimation error. As shown in Theorem 2, new sufficient condition under certain assumptions is given to achieve the desired performance requirement and an upper bound of the bias of developed estimation is presented simultaneously. Further research directions include the developments with respect to the global convergence criterion of the joint state and fault estimation algorithm.*

## 4 An Illustrative Example

In this section, we provide a simulation example to illustrate the feasibility and applicability of the newly proposed estimation algorithm.

Following [10,15,24], we consider the joint recursive robust state/fault estimation problem for a ballistic object tracking system. When tracking a ballistic object, the measurements are collected sequentially by a radar system equipped with a set of sensors communicating through a (possibly wireless) network. The phenomena of sensor saturations might occur due to the physical constraints. Moreover, the system may suffer from the fault owing to the unpredictable changes of the network conditions, and the ROFs may stem from the issues concerning on the aging, temporary failure of the sensors/actuators, and electromagnetic interference. In addition, the imperfections of the mathematical model are reflected by the norm-bounded uncertainties. Our objective is, therefore, to propose a fault estimation algorithm for a class of time-varying systems with parameter uncertainties, nonlinearities and sensor saturations. To this end, we are ready to design a recursive estimator such that, for all parameter uncertainties, ROFs and sensor saturations, the upper bound of the estimation error covariance can be minimized by properly designing the estimator parameters at each time step. In particular, the target abscissa, target ordinate and the fault are estimated jointly. The addressed ballistic object tracking systems are described by the following discrete-time nonlinear stochastic equations:

$$\begin{cases} \vec{x}_{k+1} = (\vec{A}_k + \Delta\vec{A}_k)\vec{x}_k + \vec{g}(\vec{x}_k) + \alpha_k \vec{F}_k f_k + \omega_k \\ \quad \vec{y}_k = \sigma(\vec{C}_k \vec{x}_k) + \nu_k \end{cases}$$

with

$$\vec{A}_k = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix},$$

$$C_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ -g \end{bmatrix}, \quad \mathcal{F}_k = \sin(3k),$$

$$\vec{F}_k = \begin{bmatrix} 0.01 \\ 0 \\ 0.01 \\ 0 \end{bmatrix}, \quad \vec{M}_k = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.2 \\ 0.1 \end{bmatrix}, \quad \vec{N}_k^T = \begin{bmatrix} 0.1 \\ 0 \\ 0.2 \\ 0.1 \end{bmatrix},$$

$$\vec{g}(\vec{x}_k) = G(h(\vec{x}_k) + H),$$

$$h(\vec{x}_k) = -\frac{g\rho(\vec{x}_{2,k})}{2\beta} \sqrt{\dot{\vec{x}}_{1,k}^2 + \dot{\vec{x}}_{2,k}^2} \begin{bmatrix} \dot{\vec{x}}_{1,k} \\ \dot{\vec{x}}_{2,k} \end{bmatrix},$$

$$\rho(\vec{x}_{2,k}) = \theta_1 \cdot \exp(-\theta_2 \vec{x}_{2,k}),$$

where $\vec{x}_k = \begin{bmatrix} \vec{x}_{1,k} & \dot{\vec{x}}_{1,k} & \vec{x}_{2,k} & \dot{\vec{x}}_{2,k} \end{bmatrix}^T$ is the state vector, $\vec{x}_{1,k}$ denotes the target abscissa, $\vec{x}_{2,k}$ represents the target ordinate, $T$ is the sampling period, $g$ is the gravity acceleration, $\beta$ is the ballistic coefficient which depends on the object mass, the shape as well as the cross-section area. The function $\rho(\cdot)$ denotes the air density, typically being an exponentially decaying function of the object height ($\theta_1 = 1.227$, $\theta_2 = 1.093 \times 10^{-4}$ for the object height $\vec{x}_{2,k} < 9144$m, and $\theta_1 = 1.754$, $\theta_2 = 1.49 \times 10^{-4}$ for the object height $\vec{x}_{2,k} \geq 9144$m). $\omega_k \in \mathbb{R}^4$ and $\nu_k \in \mathbb{R}^2$ are the zero-mean Gaussian white noises with covariances $Q_k$ and $R_k = 10I_2$. Here,

$$Q_k = c \cdot \operatorname{diag}\{\bar{Q}, \bar{Q}\}, \quad \bar{Q} = \begin{bmatrix} \frac{T^3}{3} & \frac{T^2}{2} \\ \frac{T^2}{2} & T \end{bmatrix}.$$

In the simulation, the other parameters are chosen as $g = 9.81\text{m/s}^2$, $\beta = 4 \times 10^4\text{kg/ms}^2$, $c = 0.1\text{m}^2/\text{s}^3$, $T = 1$s, $\bar{\vec{x}}_0 = 10^2 \times \begin{bmatrix} 300 & 4 & 90 & 3 \end{bmatrix}^T$, $A_{fk} = 2\sin(k)$, $\bar{\alpha} = 0.95$, $\gamma_{1,k} = \gamma_{2,k} = 0.3$, $\gamma_{3,k} = \gamma_{4,k} = 0.5$, and $\gamma_{5,k+1} = \gamma_{6,k+1} = \gamma_{7,k+1} = \gamma_{8,k+1} = 0.3$. In addition, the saturation level is set to be $\varrho_i = 33000$ $(i = 1, 2)$. By employing the Theorem 1, the simulation results can be obtained as in Figs. 1-3. Among them, the target abscissa, target ordinate and its estimations are depicted in Figs. 1-2 respectively. From the Figs. 1-3, we can see that the proposed estimation algorithm can estimate the additive fault, target abscissa and target ordinate well, which further shows the practical applicability of the developed fault estimation approach. Moreover, it is easy to see that new estimation method has better performance than the classical Kalman filter from Figs. 1-2.

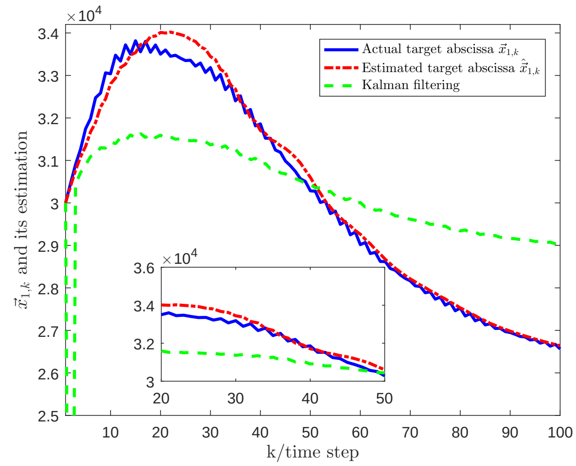


Fig. 1. The actual target abscissa and its estimation

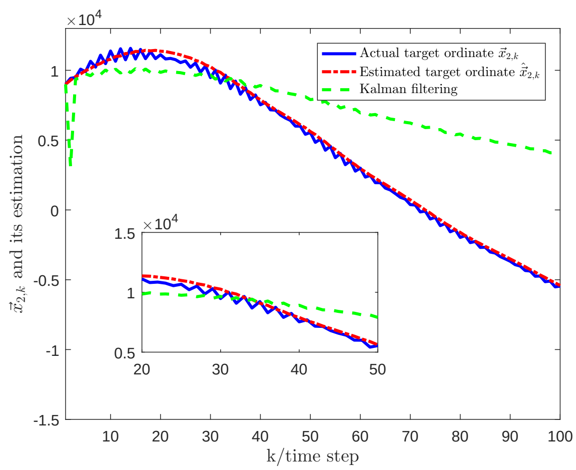


Fig. 2. The actual target ordinate and its estimation

On the other hand, for comparison, consider two cases of saturation level, i.e., $\varrho_i = 10000$ $(i = 1, 2)$ for Case I, and $\varrho_i = 5000$ $(i = 1, 2)$ for Case II. The initial condition is $\bar{\vec{x}}_0 = 10^2 \times \begin{bmatrix} 100 & 2 & 120 & 3 \end{bmatrix}^T$, and the other parameters are same as mentioned above. Again, by using Theorem 1, the desired estimator parameter can be obtained recursively and the related simulation results can be given in Figs. 4-7. As expected, it can be concluded that the algorithm accuracy is better when the saturation level is bigger.

## 5 Conclusions

In this paper, we have investigated the fault estimation problem for a class of uncertain time-varying stochastic systems with randomly occurring fault and sensor saturations. A Bernoulli random variable with known conditional probability has been employed to characterize

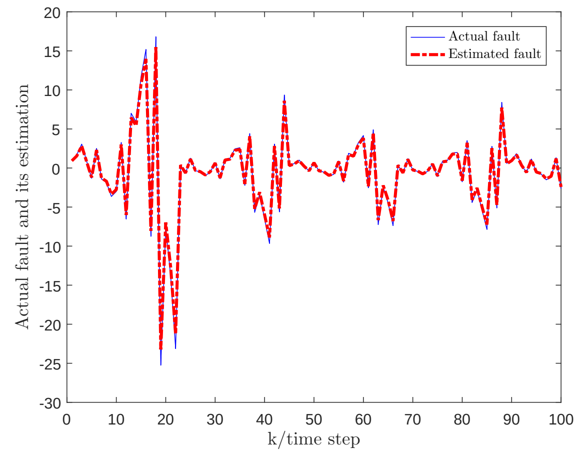


Fig. 3. The actual fault and its estimation

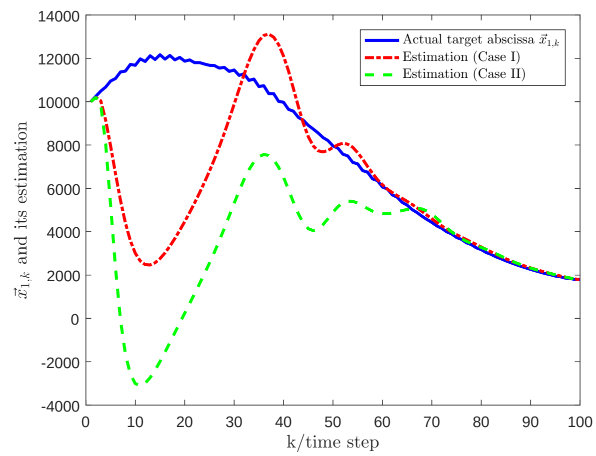


Fig. 4. The actual target abscissa and its estimation

the phenomenon of randomly occurring fault. A novel compensation scheme, which has fully taken the occurrence probability of the randomly occurring fault into account, has been given to attenuate the effects from both randomly occurring fault and sensor saturations onto the estimation performance. An optimized estimation scheme has been proposed where an upper bound of the estimation error covariance has been obtained and minimized at each sampling instant by designing the estimator gain. Moreover, a sufficient condition has been given to ensure that the estimation error is exponentially bounded in the mean square sense. Finally, simulation examples have been provided to illustrate the feasibility and effectiveness of the developed joint estimation algorithm.

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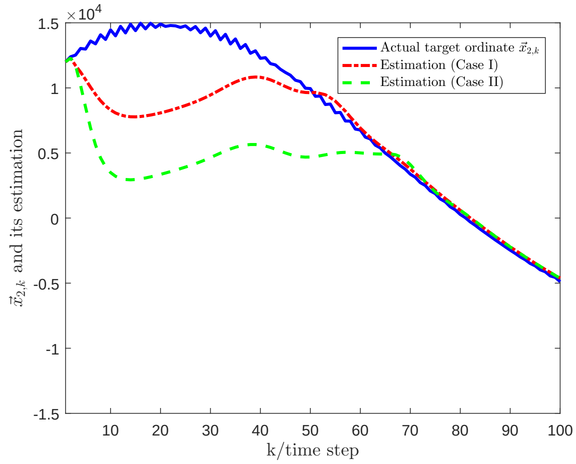


Fig. 5. The actual target ordinate and its estimation

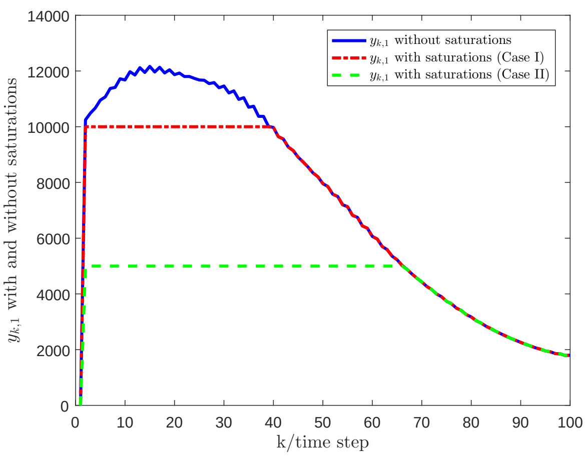


Fig. 6. $y_{k,1}$ without and with saturations (Cases I-II)

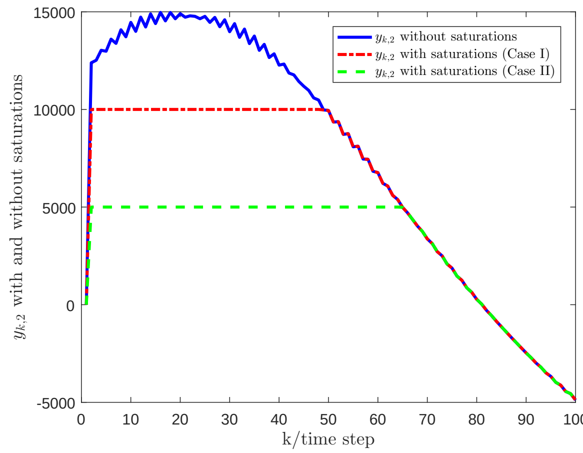


Fig. 7. $y_{k,2}$ without and with saturations (Cases I-II)

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