Can a Tromino be Tiled with Unit Trominoes?

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Abstract

Polominoes are well-known due to their use in the game Tetris, in which shapes made from four squares called tetrominoes are arranged within a game area. Polominoes can be constructed using any number of squares. In this article trominoes, which consist of three squares in an L-shape formation, are examined. We determine whether these can be used to fill larger L-shaped formations.

1 Introduction

A tromino (also known as a triomino) is a geometric shape formed by three squares. Trominoes can be either 'I' shaped or 'L' shaped and can be rotated in any orientation. In this article only the 'L' shaped tromino is used and throughout we use the word tromino to mean an 'L' shaped tromino. The possible orientations of the tromino are given in Figure 1.

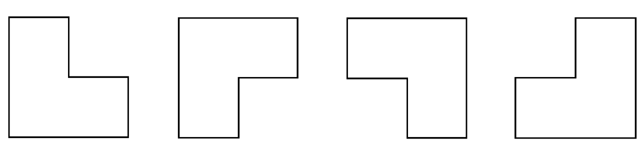

Figure 1: A unit Tromino and its Rotations

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Larger L-shapes can also be formed as shown in Figure 2 and the aim of this article is to show that such shapes can be completely filled with copies of the unit trominoes given in Figure 1. A regular L-shaped tromino can be considered to comprise three $n \times n$ squares in the arrangement given by the dotted lines in Figure 2.

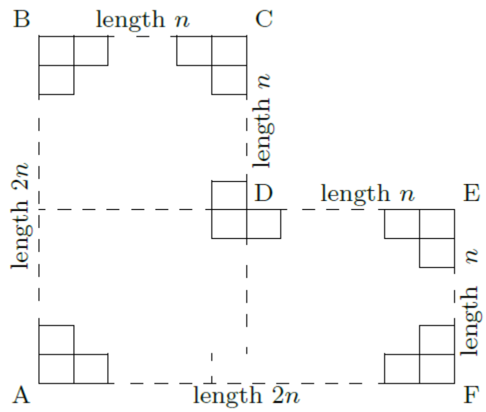
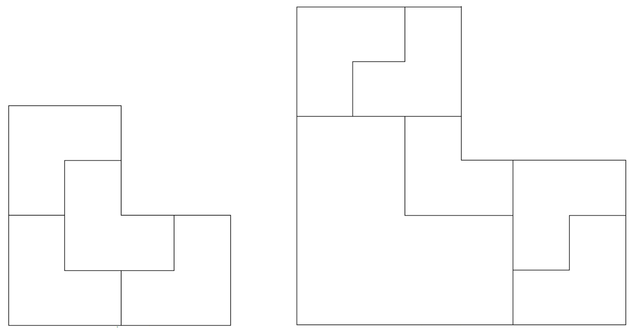


Figure 2: A Tromino of Size n

The unit trominoes in Figure 1 are denoted L_1 and the larger L-shaped tromino, in Figure 2, of size n is denoted by L_n. The aim of this article is to show that L_n for any integer n can be tiled completely with unit trominoes (L_1).

2 Tiling L_n for n a multiple of 2 or 3

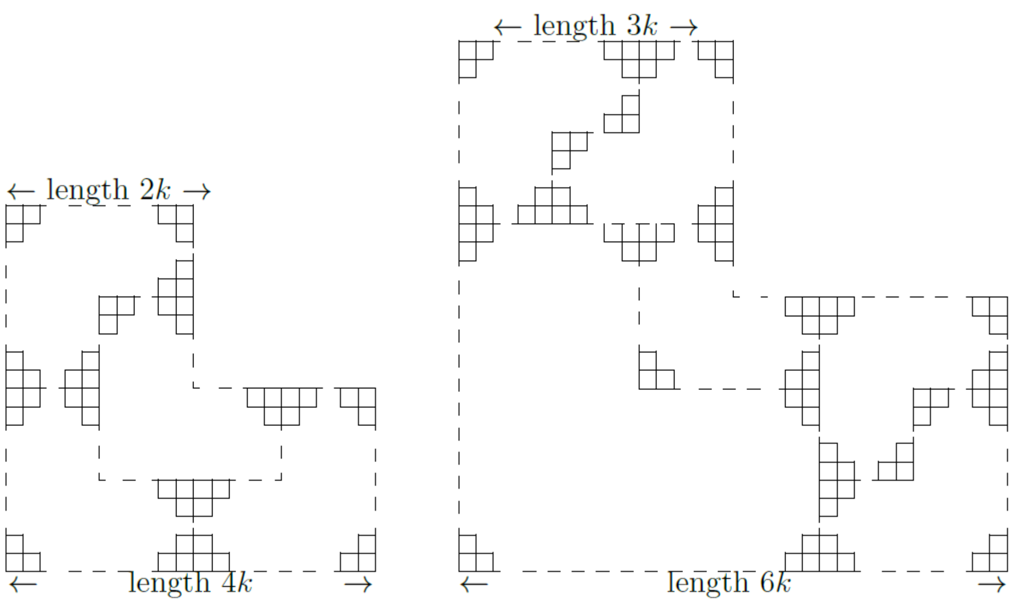
For the cases where n is a multiple of 2 or 3 it can be shown explicitly that a tiling exists. The smallest case L_2 is given in Figure 3(a) using four copies of L_1. L_3 is given in Figure 3(b) and uses five copies of L_1 and one L_2 (which is itself composed of four copies of L_1).



(a) A Tiling of L_2 (b) A Tiling of L_3

Figure 3: A Tiling of L_2 and L_3

In Figure 4(a) it is demonstrated that the same arrangement as L_2 can be used to tile L_n when n is a multiple of 2 ($n = 2k$, for k an integer) using four copies of L_k. Similarly in Figure 4(b) the same arrangement as L_3 can be used to tile L_n when n is a multiple of 3 ($n = 3k$, for k an integer) using five copies of L_k and one copy of L_{2k}.



(a) A General L_{2k} (b) A General L_{3k}

Figure 4: A General L_{2k} and L_{3k}

Therefore if there exists a tiling of a tromino L_k, of size k, then there exists a tiling of a tromino L_{2k}, of size $2k$ and there exists a tiling of a tromino L_{3k} of size $3k$.

Hence any L_n for $n = 2^x 3^y m$ can be tiled using unit trominoes (L_1) if L_m can be tiled using unit trominoes. Therefore it is sufficient to prove that L_m can be tiled by L_1 when m is not a multiple of 2 or 3.

3 Tiling of L_n, for $n \geq 5$

Consider L_n in Figure 5. First tile the squares marked x using an L_1. The remaining tromino consists of three $n \times n$ squares with a corner square removed, called **deficient squares**. To give a tiling for a tromino for L_n, for $n \geq 5$ and n not a multiple of 2 or 3, it suffices to show that an $n \times n$ deficient square (for $n \geq 5$ and not a multiple of 2 or 3) can be tiled with multiple copies of L_1.

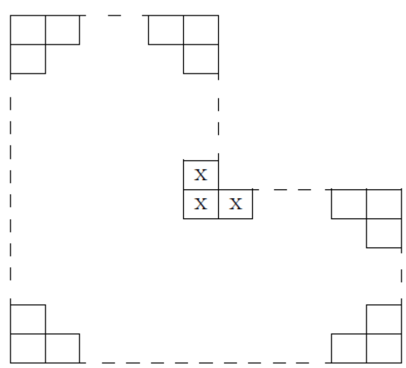
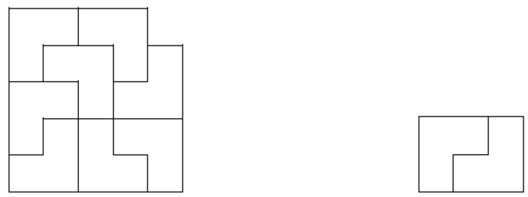


Figure 5: A Tromino Constructed from Deficient Squares

4 Tiling a Deficient $n \times n$ Square

Chu and Johnsonbaugh [**?**] show that a deficient square of dimension $n \times n$ can be tiled with unit trominoes if $n \geq 5$ and 3 does not divide n. They consider both n odd and n even, and with the removed square located anywhere within the $n \times n$ square. However only the case of n odd and the corner square removed is of interest in this article and is summarized in this section.

Chu and Johnsonbaugh [?] give an explicit tiling of a deficient 5×5 square, an example of which is given in Figure 6(a). They also give a 2×3 configuration (as shown in Figure 6(b)), which is used to show that a $2i \times 3j$, or $3i \times 2j$ board can be tiled exactly using unit trominoes (using an $i \times j$ arrangement of 2×3 resp. 3×2 configurations).

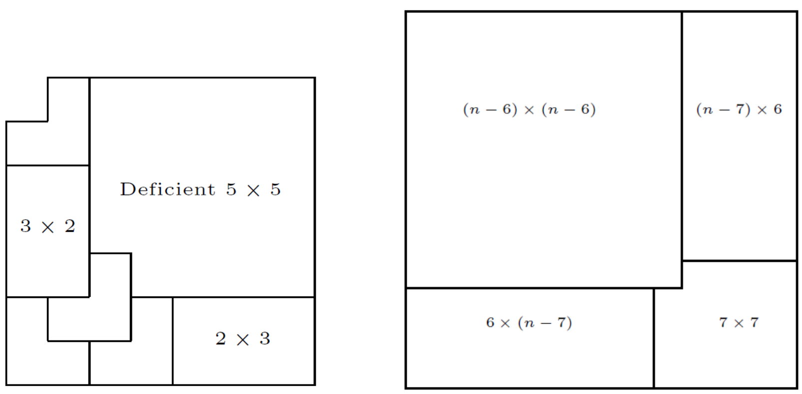


(a) A 5×5 Deficient Board Tiled Using Trominoes

(b) A 2×3 Configuration

Figure 6: 5×5 and 2×3 Configurations

Using the 5×5 deficient square and the 2×3 and 3×2 configurations, the 7×7 deficient square can now be tiled (Figure 7(a)).



(a) A 7×7 deficient square

(b) A $n \times n$ deficient square for n odd

Figure 7: A 7×7 and $n \times n$ (for n odd) deficient square

A $n \times n$ deficient square for $n \geq 11$ and n odd is given in Figure 7(b) and comprises of four shapes. This is a rearranged version of Figure 9 given in [?] using the same shapes but with a corner square removed. Consider each of these shapes in turn:

- $(n-7) \times 6$ (and hence also $6 \times (n-7)$) rectangle can be tiled using the configurations given in Figure

6(b).

- 7×7 deficient square is given in Figure 7(a).

- $(n-6) \times (n-6)$ deficient square, the first case when $(n-6) \times (n-6) = 5 \times 5$ is given in Figure 6(a) and when $(n-6) \times (n-6) = 7 \times 7$ is given in Figure 7(a). Therefore since an 11×11 deficient square can be created when $(n-6) \times (n-6) = 5 \times 5$ and 13×13 is created using $(n-6) \times (n-6) = 7 \times 7$ then by an inductive argument all remaining cases for $n = 5+6k$, $7+6k$, i.e. n odd and not a multiple of three can be generated.

5 A tiling of L_n

In Section 2 it was demonstrated that L_n can be tiled using unit trominoes (L_1) for n a multiple of 2 or 3. The results of Chu and Johnsonbaugh [**?**] demonstrate that an $n \times n$ deficient square can be tiled using unit trominoes for $n \geq 5$, n odd and where 3 does not divide n. Since the deficient square can be chosen to be a corner square then three such squares plus one copy of L_1 can be arranged as shown in Figure 5 to construct L_n where $n \geq 5$, n odd and where 3 does not divide n. Hence it has been shown that there exists a tiling of L_n, for any integer n using unit trominoes.

References

[1] I. Chu and R. Johnsonbaugh, *Tiling deficient boards with trominoes*, Mathematics Magazine **59** (1986), no. 1, 34–40.