Determining the kinematic properties of an advancing front using a decomposition method

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***Abstract*—In applications where partial differential equations are used to model populations, there is frequently a critical density threshold below which the population cannot be detected in practice and the corresponding position is often termed the leading edge of the distribution. Historically this position has been investigated for large time problems, but little attention has been afforded to understanding its short term dynamics. In this work we describe a novel approach, utilizing the Laplace decomposition method, that generates algebraic expressions for the initial kinematic properties of the leading edge in terms of the initial data and model parameters. The method is demonstrated on two well-studied partial differential equations and two established systems of equations (representing the growth of fungal networks), all of which display travelling fronts. The kinematics of these advancing fronts are determined using our method and are shown to be in excellent agreement with both exact solutions and numerical approximations of the model equations.**

***Index Terms*—Laplace decomposition method, partial differential equations, travelling wave, numerical solution, fungi.**

I. Introduction

MANY systems of partial differential equations (PDEs) display travelling wave solutions. Traditionally, and especially in a single spatial dimension, the development of such systems is investigated analytically, e.g. by using substitutions of the form $z = x - ct$. However, for highly non-linear equations, or complicated initial data, it may be too complex to extract such solutions, even if they exist. While numerical simulations can provide insight into the behaviour of the system [1], parameter values and initial data have to be chosen in advance and it is therefore often difficult to isolate the influence of either on the entire system. Furthermore, such solutions may only describe the long term behaviour and not the initial development of the system which is influenced by initial data [2] and can be especially important when considering population dynamics. For example, in infectious diseases, the early dynamics of the disease in a spatially-organized population is determined by its initial distribution [3], in ecology the progress of non-native species invading an otherwise empty environment is of interest [4], and in medicine the progress of a drug towards a tumour is dependent on the initial concentration of the drug [5].

An additional problem faced in applications when using PDEs is that the equations can predict the presence of a population at low densities that are, in practice, impossible to measure in the physical system being considered. Thus the tracking of a leading edge of the population, namely the position at which a population can be first detected, is crucial to allow comparison between theoretical and experimental results. In this work an analytical approach is proposed that can be used to describe the behaviour of such leading edges by constructing algebraic expressions for their initial kinematics. This novel approach, described in Section II, is based on constructing a series solution of the model equations utilizing the Laplace decomposition method (LDM) and allows the influence of the initial data and all model parameters to be easily observed. In Section III the method is applied to an advection equation with spatially-dependent velocity and the Kawahara equation. An alternative practical application is then considered in Section IV by using the method on a class of systems of PDEs constructed by Edelstein-Keshet [6] to describe the growth of fungal colonies. In both cases, the method output is compared to analytical and numerical solutions.

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II. Kinematics of the leading edge of a wave front

A. Formulation

Suppose $u(x,t)$ satisfies $u_t = f(x,t,u,u_x,u_{xx},\ldots)$ for a given function f with positive initial data $u(x,0) = u_0(x)$ representing an "invasive" population, i.e. $u_0(x) \to 0$ as $x \to \infty$. To track how the distribution of u develops over time, suppose x_0 is the greatest value of x such that $u(x_0,t_0) = u_c$ for some arbitrary u_c at time t_0 taken to be in the range of u and that satisfies $u_x(x_0,t_0) < 0$. The value u_c, denoting the leading edge of the distribution, can be best regarded as representing a critical density below which the population cannot be detected in practice. After a time Δt has elapsed, suppose the leading edge has moved a distance Δx so that $u(x_0+\Delta x, t_0+\Delta t) = u_c$.

Since the position Δx is a function of time Δt we suppose $\Delta x = \sum_{k=0}^{\infty} a_k \Delta t^k$. Clearly $a_0 = 0$ while a_1 and $2a_2$ correspond to the velocity and acceleration respectively at time t_0. By taking a Taylor series of $u(x_0+\Delta x, t_0+\Delta t)$, setting both $u(x_0+\Delta x, t_0+\Delta t)$ and $u(x_0,t_0)$ equal to u_c and using $\Delta x = \sum_{k=0}^{\infty} a_k \Delta t^k$, it follows that

$$0 = \Delta t\left[a_1 u_x + u_t\right] + \Delta t^2\left[a_2 u_x + \frac{a_1^2}{2}u_{xx} + a_1 u_{xt} + \frac{1}{2}u_{tt}\right] + O(\Delta t^3).$$

Neglecting higher order terms and comparing coefficients allows a_1 and a_2 to be determined and hence the instantaneous velocity of the wave front at (x_0,t_0) is $-u_t/u_x$ while the acceleration is $(2u_t u_x u_{xt} - u_{tt}u_x^2 - u_t^2 u_{xx})/u_x^3$, provided u_x is non-zero.

While these formulae provide simple estimates of the velocity and acceleration of the leading edge of an advancing wave front, they require the solution of the corresponding PDE to be known. If such a solution is unknown, a series solution can be obtained using the Laplace decomposition method to generate corresponding formulae in terms of the original model parameters.

B. Laplace Decomposition Method

For a given PDE $u_t = f(x, t, u, u_x, u_{xx}, \ldots)$ a series solution for $u(x, t)$ is assumed, i.e. $u(x, t) = \sum_{m=0}^{\infty} u_m(x, t)$. By taking Laplace transforms of this series representation, iterative formulae are constructed relating each term in the series to previous terms and where the first term is constructed from the initial data (see [7], [8], [9] for further details). The Laplace transform of nonlinear terms are represented as Adomian polynomials, which are essentially series representations constructed from the previously calculated terms in the series for $u(x, t)$ (see [10], [11], [12], [13]). This method, termed the Laplace decomposition method (LDM), has been widely used to investigate an array of nonlinear differential equations (e.g. [9], [14], [15], [16], [17], [18]). The convergence of the series can be proven (e.g. [19], [20], [21], [22]) and, furthermore, the series corresponds to a Taylor series of a closed form solution, should one exist. The resultant series can therefore be written as

$$u(x, t) = u_0(x) + u_1(x)t + u_2(x)t^2 + \ldots$$

and hence the initial velocity of the leading edge of the wave front (i.e. when $t = 0$) at x_0 is

$$-\frac{u_t}{u_x} = -\frac{u_1}{u_0'} \tag{1}$$

and the initial acceleration is

$$\frac{2u_t u_x u_{xt} - u_{tt}u_x^2 - u_t^2 u_{xx}}{u_x^3} = \frac{2u_0' u_1 u_1' - 2u_2 (u_0')^2 - u_1^2 u_0''}{(u_0')^3}, \tag{2}$$

where prime denotes differentiation with respect to x and the functions are evaluated at x_0.

III. EXAMPLES

A. Advection equation with non-constant velocity

Consider the advection equation with spatially-dependent velocity $c(x)$

$$u_t = -\left(c(x)u\right)_x \tag{3}$$

and specified initial data $u(x, 0)$. Applying Laplace transforms to equation (3), rearranging and taking inverse Laplace transforms yields

$$u(x, t) = u(x, 0) + \mathcal{L}^{-1}\left[-\frac{\mathcal{L}[(c(x)u)']}{s}\right]. \tag{4}$$

Assuming the series solution is of the form $u(x, t) = \sum_{m=0}^{\infty} u_m(x, t)$, then equation (4) becomes

$$\sum_{m=0}^{\infty} u_m(x, t) = u(x, 0) + \sum_{m=0}^{\infty} \mathcal{L}^{-1}\left[\frac{-\mathcal{L}[(c(x)u_m)']}{s}\right]$$

giving the iterative formula

$$\begin{aligned} u_0(x, t) &= u(x, 0), \\ u_{m+1}(x, t) &= -\mathcal{L}^{-1}\left[\frac{\mathcal{L}[(c(x)u_m)']}{s}\right]. \end{aligned} \tag{5}$$

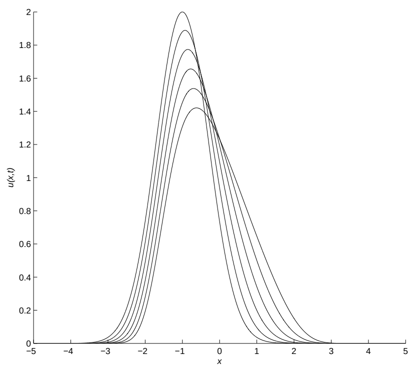

Fig. 1. Equation (3) is solved numerically with initial data $u(x, 0) = 2e^{-(x+1)^2}$ and $c(x) = 1 + \epsilon \sin x$ where $\epsilon = 0.5$. The solutions for $u(x, t)$ are shown at times at times $t = 0, 0.2, 0.4, 0.6, 0.8$ and 1.0. The distributions propagate from left to right as t increases.

For illustrative purposes, suppose the local velocity and initial data are $c(x) = 1 + \epsilon \sin(x)$ and $u(x, 0) = 2e^{-(x+1)^2}$ respectively. The resultant series constructed from (5) is then

$$\begin{aligned} u(x, t) = e^{-(x+1)^2}\Big[& 2 + \Big\{4 + 4x \\ & -2\epsilon\left(\cos x - 2(x+1)\sin x\right)\Big\} t + \Big\{2 + 8x + 4x^2 \\ & +\epsilon\left((3 + 16x + 8x^2)\sin x - 6(x+1)\cos x\right) \\ & +\epsilon^2\left(1 + 4x(x+2)\sin^2 x - 6(x+1)\sin x \cos x\right)\Big\} t^2 \Big] \\ & + \cdots \end{aligned} \tag{6}$$

Suppose the critical value of the leading edge is taken to be $u_c = 2e^{-1}$ so that $x_0 = 0$. The initial velocity and acceleration of the wave front as calculated by equation (1) and (2) are respectively

$$1 - \frac{\epsilon}{2}, \quad \text{and} \quad \epsilon - \frac{5}{4}\epsilon. \tag{7}$$

When $\epsilon = 0$ equation (3) has a travelling wave solution $u(x, t) = 2e^{-(x-t+1)^2}$ corresponding to the initial data propagating to the right with a constant unit velocity, consistent with (7). For $\epsilon > 0$ numerical solutions of (3) were obtained and demonstrate the initial distribution also propagates to the right but at a spatially-dependent rate (Fig. 1). These numerically obtained distributions are in good qualitative and quantitative agreement with the analytical approximation constructed in equation (6) (Fig. 2). Furthermore, the initial velocities and accelerations of wave fronts at the above critical value of u_c were calculated numerically for various ϵ and were compared to the predictions from (7) (Table I).

B. Kawahara equation

The Kawahara equation

$$u_t + \alpha u u_x + \beta u_{xxx} - \mu u_{xxxxx} = 0 \tag{8}$$

arises in the investigation of magneto-acoustic waves in plasmas [23] and in shallow water waves [24]. The equation is known to exhibit a travelling wave solution in terms

TABLE I
EQUATION (3) WAS SOLVED NUMERICALLY WITH INITIAL DATA $u(x,0) = 2e^{-(x+1)^2}$ AND $c(x) = 1 + \epsilon \sin x$ AND THE VELOCITIES AND ACCELERATIONS OF THE LEADING EDGE OF THE WAVE FRONTS STARTING AT $x_0 = 0$ WERE CALCULATED AT TIME $t = 10^{-4}$. THE INITIAL VELOCITY AND ACCELERATION OF THE LEADING EDGE OBTAINED ANALYTICALLY FROM (7) ARE SHOWN FOR COMPARISON.

Parameter	Wave front velocity		Wave front acceleration	
ϵ	Equation (7)	Numerical solution of (3)	Equation (7)	Numerical solution of (3)
0	1	1.0000	0	0.0006
0.5	0.75	0.7500	0.1875	0.1877
1.0	0.5	0.4999	-0.25	-0.2503
5.0	-1.5	-1.5028	-26.25	-26.2710

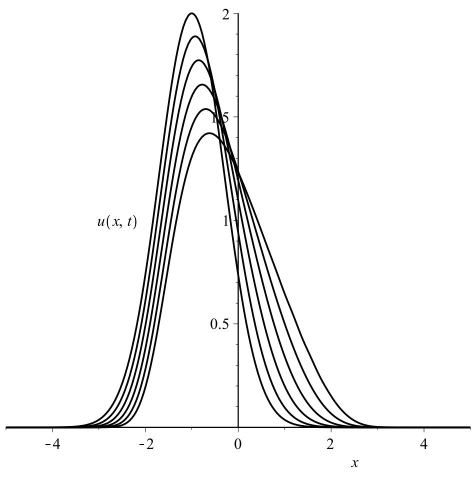

Fig. 2. The first thirty terms in the series solution from equation (6) are used to produce distributions of $u(x,t)$ at times $t = 0, 0.2, 0.4, 0.6, 0.8$ and 1.0 where $\epsilon = 0.5$. These distributions propagate from left to right as t increases (cf. numerical solutions in Fig. 1).

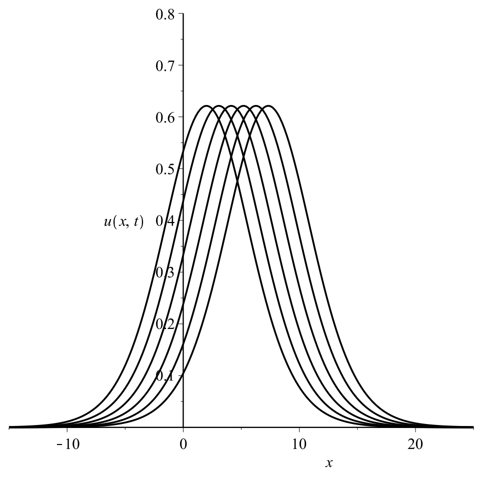

Fig. 3. Equation (10), representing the solution of equation (8) with initial data (9), $\alpha = \beta = \mu = 1$ and $\hat{x} = 2$ is shown at times $t = 0, 5, 10, 15, 20$ and 25. The distribution propagates from left to right.

of sech^4 and the corresponding velocity can be calculated exactly [25]. Hence suppose the initial data is given by

$$u(x,0) = \frac{105}{169}\text{sech}^4\Phi \tag{9}$$

where $\Phi = \frac{x-\hat{x}}{2\sqrt{13}}$ and $\hat{x}$ is the location of the maximum value of the distribution. The analytical solution of equation (8) with $\alpha = \beta = \mu = 1$ is

$$u(x,t) = \frac{105}{169}\text{sech}^4\left(\frac{1}{2\sqrt{13}}\left(x - \frac{36}{169}t - \hat{x}\right)\right), \tag{10}$$

which may be easily verified through substitution, and has corresponding wave speed $\frac{36}{169}$ (Fig. 3).

Taking Laplace transforms of equation (8), rearranging and taking inverse Laplace transforms yields

$$u(x,t) = u(x,0) + \mathcal{L}^{-1}\left[\frac{\mathcal{L}\left[\mu u_{xxxxx} - \beta u_{xxx} - \alpha u u_x\right]}{s}\right] \tag{11}$$

Assuming the series solution is of the form $u(x,t) = \sum_{m=0}^{\infty} u_m(x,t)$, equation (11) yields

$$\sum_{m=0}^{\infty} u_m(x,t) = u(x,0) + \sum_{m=0}^{\infty}\mathcal{L}^{-1}\left[\frac{\mathcal{L}\left[\mu u^{(5)} - \beta u''' - A_m\right]}{s}\right]$$

where the nonlinear term $\alpha u u_x$ is decomposed into the Adomian polynomials A_m defined by

$$A_m = \alpha \sum_{i=0}^{k} u_i u'_{m-i}.$$

The iterative formula for $u(x,t)$ is therefore

$$u_0(x,t) = u(x,0)$$
$$u_{m+1}(x,t) = \mathcal{L}^{-1}\left[\frac{\mathcal{L}\left[\mu u^{(5)} - \beta u''' - A_m\right]}{s}\right], \ m \geq 0$$

and with initial data (9) yields the following series

$$\begin{aligned} u(x,t) = {} & \frac{105}{169}\,\text{sech}^4\Phi \\ & + \frac{105\sqrt{13}}{371293}\text{sech}^4\Phi\Big[210(\alpha-\mu)\tanh^5\Phi \\ & + 15(28\alpha + 13\beta + 15\mu)\tanh^3\Phi \\ & + (210\alpha - 91\beta - 47\mu)\tanh\Phi\Big]t + \cdots \end{aligned} \tag{12}$$

Suppose the leading edge of the distribution corresponds to the rightmost position where $u(x,t) = 0.01$. At $t = 0$ and setting $\hat{x} = 2$, the corresponding solution of equation (9) gives $x_0 = 14.202$ (Fig. 3). Substituting x_0 into equations (1) and (2) and evaluating the terms in (12) gives the instantaneous velocity and acceleration of the leading edge as

$$\begin{aligned} & 0.0100\alpha + 0.2345\beta - 0.03148\mu \qquad \text{and} \\ & 0.0075\alpha\beta - 0.0075\alpha\mu - 0.0016\mu^2 + 0.0182\mu\beta - 0.0166\beta^2 \end{aligned} \tag{13}$$

respectively. When $\alpha = \beta = \mu = 1$, (13) approximates the initial wave speed at the leading edge as 0.213, which agrees with the known velocity of $\frac{36}{169}$, and the acceleration is zero, consistent with the constant velocity of the solution.

In addition to approximating the initial kinematic properties of the leading edge, equation (12) provides an alternative means of tracking the position of the leading edge for larger times. To this end, truncated series of equation (12) were constructed and solved to determine the position of the leading edge of the distribution (Table II). The accuracy of the position of the leading edge obtained from the series decreases with time but improves with the inclusion of more terms, similar to a Taylor series approximation of a function.

IV. Applications

Edelstein-Keshet [6] proposed a series of mathematical models comprising systems of nonlinear PDEs to represent the growth of a fungal network. These models correspond to different morphological types of fungi colonizing a region by a propagating travelling wave, and where the existence of the solutions were proven by phase plane methods. Indeed, while Edelstein-Keshet's models have been refined and further developed in recent years [26], [27], [28], [29], these newer models all retain similar components.

A. Fungal growth model: decoupled equations

One morphology considered in [6] is represented by the pair of equations

$$\begin{aligned} \rho_t &= vn - \gamma\rho, \\ n_t &= -vn_x + \alpha n - \beta n^2, \end{aligned} \tag{14}$$

where $\rho = \rho(x,t)$ and $n = n(x,t)$ denote the density of fungal material and v, α, β and γ are positive parameters related to movement, creation, and loss of material. Since the equations are decoupled, the second equation in (14) for $n(x,t)$ can be considered in isolation.

Suppose the initial data for $n(x,t)$ is of the form

$$n(x,0) = \frac{\alpha}{\beta}\left[\frac{1-\tanh(\phi x)}{2}\right], \tag{15}$$

representing the case where there is no fungi on the right but a steady state density on the left, corresponding to the equilibria determined from equation (14). While not discussed in [6], equation (14) has a closed form solution for $n(x,t)$, which with initial data (15) can be shown to be

$$n(x,t) = \frac{\alpha}{2\beta}\left(1-\tanh\left(\phi\left(x-\left[v+\frac{\alpha}{2\phi}\right]t\right)\right)\right).$$

Clearly this is a travelling wave solution with wave speed $v + \alpha/(2\phi)$. The series solution of equation (14) with initial data (15) will now be constructed and the speed and acceleration of the leading edge of the wave front calculated.

Applying Laplace transforms to the second equation in (14), rearranging and then taking inverse transforms gives

$$n(x,t) = n(x,0) + \mathcal{L}^{-1}\left[\frac{\mathcal{L}[-vn' + \alpha n - \beta n^2]}{s}\right], \tag{16}$$

where $n(x,0)$ is specified initial data. Assuming the series solution for $n(x,t)$ is of the form $n(x,t) = \sum_{m=0}^{\infty} n_m(x,t)$, then (16) yields

$$\sum_{m=0}^{\infty} n_m(x,t) = n(x,0) + \sum_{m=0}^{\infty} \mathcal{L}^{-1}\left[\frac{\mathcal{L}[-vn'_m + \alpha n_m - A_m]}{s}\right],$$

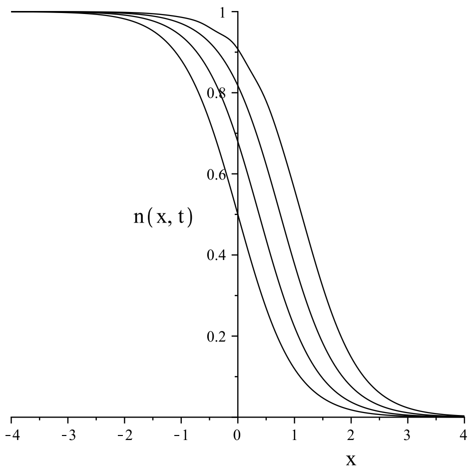

Fig. 4. The first twelve terms in the series solution from equation (18) are used to produce distributions of $n(x,t)$ at times $t = 0, 0.25, 0.5$ and 0.75. All model parameters are set to unity.

where the nonlinear term βn^2 is decomposed and the related A_m terms, i.e. the Adomian polynomials, are given by

$$A_m = \beta \sum_{i=0}^{m} n_i n_{m-i}.$$

Hence the iterative formula for constructing the terms in $n(x,t)$ is

$$\begin{aligned} n_0(x,t) &= n(x,0), \\ n_{m+1}(x,t) &= \mathcal{L}^{-1}\left[\frac{\mathcal{L}[-vn'_m + \alpha n_m - A_m]}{s}\right], \quad m \ge 0. \end{aligned} \tag{17}$$

Thus equation (17) with initial data (15) generates

$$\begin{aligned} n(x,t) =& \frac{\alpha}{\beta}\left(\frac{1-\tanh(\phi x)}{2}\right) + \frac{\alpha}{4\beta}\left(\frac{2v\phi+\alpha}{\cosh^2(\phi x)}\right)t \\ &+ \frac{\alpha}{8\beta}\left(\frac{(2v\phi+\alpha)^2\sinh(\phi x)}{\cosh^3(\phi x)}\right)t^2 + \dots \end{aligned} \tag{18}$$

and can be used to produce approximate solutions of equation (14) that display the travelling wave profile (Fig. 4). The initial velocity of the leading edge from equation (1) at $x_0 = 0$ (i.e. u_c is taken to be at the centre of the initial distribution) is

$$-\frac{n_1}{n'_0} = v + \frac{\alpha}{2\phi},$$

in agreement with the travelling wave structure of the closed form solution, and the related acceleration from equation (2) is easily shown to be zero, consistent with the constant velocity of the travelling wave.

B. Fungal growth model: coupled equations

A second model proposed by Edelstein [6] is given by

$$\begin{aligned} \rho_t &= vn - \gamma\rho, \\ n_t &= -vn_x + \alpha\rho - \beta n\rho, \end{aligned} \tag{19}$$

where the model variables and parameters are consistent with those above. These equations are coupled and have no known closed form solutions.

TABLE II

THE POSITION OF THE LEADING EDGE OF EQUATION (8) (DETERMINED BY THE GREATEST VALUE OF x WHERE $u(x,t) = 0.01$) WITH $\alpha = \beta = \mu = 1$, INITIAL DATA (9) AND $\hat{x} = 2$ IS CALCULATED USING THE ANALYTICAL SOLUTION (10) AND FROM TRUNCATED SERIES IN EQUATION (12) COMPRISING RESPECTIVELY THE FIRST 2 AND 4 TERMS.

Time	Equation (10)	Series (12): 2 terms	Series (12): 4 terms
0	14.202	14.202	14.202
2	14.628	14.588	14.628
4	15.054	14.910	15.052
6	15.480	15.187	15.473
8	15.906	15.428	15.885
10	16.332	15.643	16.287

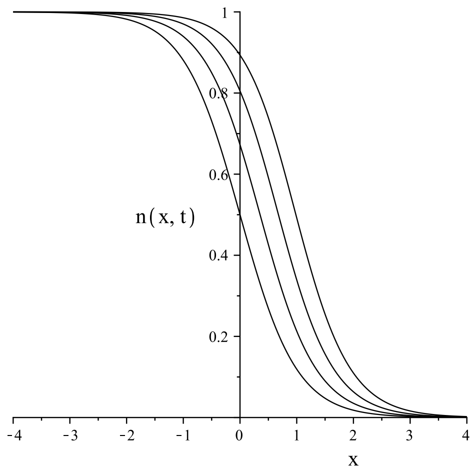

Fig. 5. The first twelve terms in the series solution from equation (23) are used to produce distributions of $n(x,t)$ at times $t = 0, 0.25, 0.5$ and 0.75. All model parameters are set to unity.

Applying Laplace transforms to equation (19), rearranging and then taking inverse transforms gives

$$\begin{aligned} \rho(x,t) &= \rho(x,0) + \mathcal{L}^{-1}\left[\frac{\mathcal{L}[vn - \gamma\rho]}{s}\right], \\ n(x,t) &= n(x,0) + \mathcal{L}^{-1}\left[\frac{\mathcal{L}[-vn' + \alpha\rho - \beta n\rho]}{s}\right]. \end{aligned} \tag{20}$$

Assuming $\rho(x,t) = \sum \rho_m(x,t)$ and $n(x,t) = \sum n_m(x,t)$, equation (20) yields

$$\begin{aligned} \sum_{m=0}^{\infty} \rho_m(x,t) &= \rho(x,0) \\ &\quad + \sum_{m=0}^{\infty} \mathcal{L}^{-1}\left[\frac{\mathcal{L}[vn_m - \gamma\rho_m]}{s}\right], \\ \sum_{m=0}^{\infty} n_m(x,t) &= n(x,0) \\ &\quad + \sum_{m=0}^{\infty} \mathcal{L}^{-1}\left[\frac{\mathcal{L}[-vn'_m + \alpha\rho_m - A_m]}{s}\right], \end{aligned}$$

where the nonlinear term $\beta n\rho$ is decomposed into the Adomian polynomials

$$A_m = \beta \sum_{i=0}^{m} n_i \rho_{m-i}.$$

Hence the iterative formula for constructing $n(x,t)$ and $\rho(x,t)$ are

$$\begin{aligned} \rho_0(x,t) &= \rho(x,0), \\ \rho_{m+1}(x,t) &= \mathcal{L}^{-1}\left[\frac{\mathcal{L}[vn_m - \gamma\rho_m]}{s}\right], \qquad m \geq 0, \\ n_0(x,t) &= n(x,0), \\ n_{m+1}(x,t) &= \mathcal{L}^{-1}\left[\frac{\mathcal{L}[-vn'_m + \alpha\rho_m - A_m]}{s}\right], \qquad m \geq 0. \end{aligned} \tag{21}$$

Consistent with earlier work, we set the initial data to be

$$\begin{aligned} \rho(x,0) &= \frac{v\alpha}{\beta\gamma}\left[\frac{1 - \tanh(\theta x)}{2}\right], \\ n(x,0) &= \frac{\alpha}{\beta}\left[\frac{1 - \tanh(\phi x)}{2}\right] \end{aligned} \tag{22}$$

representing the case where the fungus is at its steady state density as $x \to -\infty$ and absent as $x \to +\infty$. Equation (21) generates the series for $n(x,t)$ (Fig. 5):

$$\begin{aligned} n(x,t) &= \frac{\alpha}{\beta}\left(\frac{1 - \tanh(\phi x)}{2}\right) \\ &+ \left\{ \frac{v\alpha\phi}{2\beta}\operatorname{sech}^2(\phi x) \right. \\ &\quad \left. - \frac{v\alpha^2}{4\beta\gamma}\left(\tanh(\phi x) + 1\right)\left(\tanh(\theta x) - 1\right) \right\} t \\ &+ \left\{ \frac{v^2\alpha\phi^2 \tanh(\phi x)}{2\beta\cosh^2(\phi x)} \right. \\ &\quad - \frac{v^2\alpha^3\left(\tanh(\phi x) + 1\right)\left(1 - \tanh(\theta x)\right)^2}{16\beta\gamma^2} \\ &\quad - \frac{v\alpha^2\left(\tanh(\phi x) + 1\right)\left(\tanh(\phi x) - \tanh(\theta x)\right)}{8\beta} \\ &\quad + \frac{v^2\alpha^2\theta\left(\tanh(\phi x) + 1\right)}{8\gamma\beta\cosh^2(\theta x)} \\ &\quad \left. - \frac{v^2\phi\alpha^2\left(1 - \tanh(\theta x)\right)}{4\gamma\beta\cosh^2(\phi x)} \right\} t^2 + \cdots \end{aligned} \tag{23}$$

The initial velocity and acceleration of the leading edge of the wave front of $n(x,t)$ (where the critical value u_c is again taken to be at the centre of the initial distribution) calculated by equation (1) and (2) at $x_0 = 0$ are respectively

$$-\frac{n_1}{n'_0} = v + \frac{v\alpha}{2\gamma\phi} \tag{24}$$

$$\frac{2n'_0 n_1 n'_1 - 2n_2(n'_0)^2 - n_1^2 n''_0}{(n'_0)^3} = \frac{v^2\alpha\left(\alpha\phi - 2\gamma\theta\phi - 2\alpha\theta\right)}{4\gamma^2\phi^2}. \tag{25}$$

TABLE III

EQUATIONS (19) WERE SOLVED NUMERICALLY AND THE VELOCITIES AND ACCELERATIONS OF THE LEADING EDGE OF THE WAVE FRONTS STARTING AT $x = 0$ WERE CALCULATED AT THE TIMES INDICATED. THE INITIAL VELOCITY AND ACCELERATION OF THE LEADING EDGE OBTAINED ANALYTICALLY FROM EQUATIONS (24) AND (25) ARE SHOWN FOR COMPARISON.

Parameters						Wave front velocity					Wave front acceleration	
v	α	β	γ	ϕ	θ	Equation (24)	$t = 0.001$	$t = 0.01$	$t = 0.1$	$t = 1$	Equation (25)	$t = 0.0001$
1	1	1	1	1	1	1.50	1.4992	1.4927	1.4333	1.1931	-0.750	-0.74
2	3	1	2	2	2	2.75	2.7243	2.7028	2.3545	2.1466	-4.125	-4.13
4	2	3	1	2	2	6.00	5.9983	5.8959	4.7465	4.1326	-24	-24.10

Thus the wave front will initially propagate to the right and its velocity will change depending on parameter values.

As there is no known analytical solution against which to compare the above expressions for the initial velocity and acceleration of the leading edge of the wave front, equation (19) is instead investigated numerically. The model equations (19) were solved for a range of initial data (22) and parameter values and the velocities and accelerations of the advancing distributions in $n(x,t)$ were calculated at various times (Table III).

V. DISCUSSION

In many applications, the spread of a continuously-distributed population is of interest. The speed and acceleration of this spread, measured using the position of the leading edge of a moving front, is determined by initial data and model parameters. We have described a simple yet powerful method utilizing the Laplace decomposition method (LDM) that calculates the initial velocity and acceleration of this leading edge in terms of the original model parameters and initial data by constructing a series solution and have shown it to be in excellent agreement with analytical and numerical solutions (Tables I,III). This is, to the authors' knowledge, the first time that any decomposition method has been applied to generate predictions on the movement of travelling fronts. Indeed, this approach works equally well on systems of partial differential equations containing nonlinear terms and allows individual wave fronts from different model variables to be tracked independently. Furthermore, the resultant series can be used to successfully track the position of the leading edge for larger times provided a sufficient number of terms are used (Table II).

In this study attention was focussed on partial differential equations (PDEs) containing only a first order derivative of time, i.e. PDEs of the form $u_t = f(u, u_x, u_x x, \ldots)$. However, equations with higher time derivatives, e.g. the wave equation $u_{tt} = cu_{xx}$, can be dealt with in a similar manner provided suitable initial data is given so that the Laplace transforms of the model variables can be determined. The iterative formula can be constructed in a similar manner where the first terms in the series again relate to the initial data.

Boundary conditions were largely ignored when describing the application of the LDM since attention was focused on the position of the leading edge of a distribution that was assumed to be distal from a boundary. Indeed, if the initial data, which forms the first term in the series solution, satisfies a given boundary condition, the method applied in this work ensures that the resultant series also satisfies the same boundary condition as the further terms vanish as $x \to \pm\infty$. In cases where the initial data does not satisfy the boundary conditions, the iterative formula must be suitably adapted at each step to ensure that the boundary conditions remain satisfied.

The algorithmic nature of the method allows for it to be easily implemented into a computer algebra package. Using only a small number of terms from the series, it is possible to construct accurate solutions without specifying parameter values in advance. Of course, these solutions are only valid in regions where the series converges [22]. In this study attention has been focused on small time problems where such convergence is guaranteed. Clearly further work is required to adapt the methods for their use in large time problems.

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